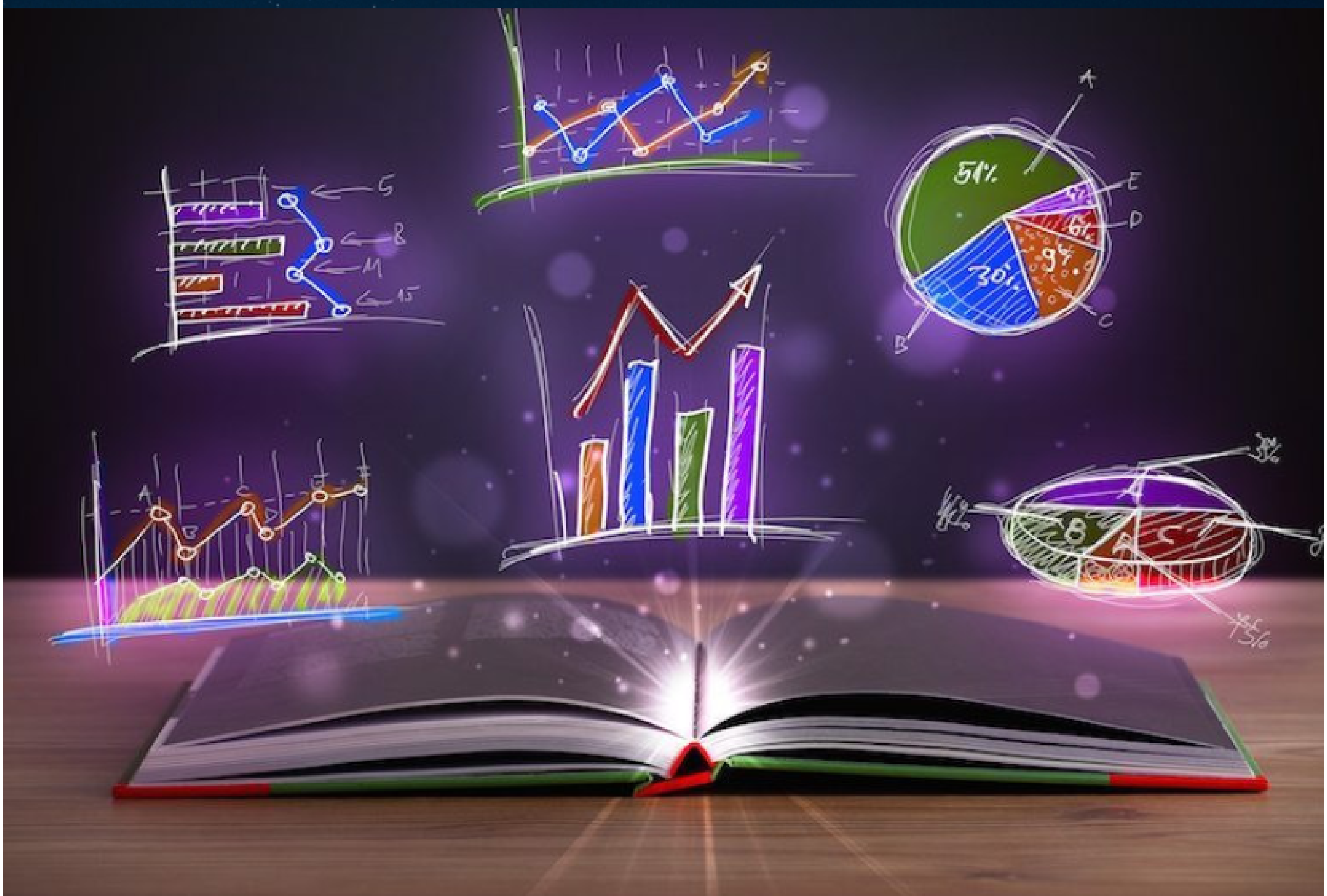
Inequality

# A brief Introduction to Inequalities

## A brief study




GAUSSIAN CURVATURE

Rama1728

Aritra Mondal

Phoenixfire

# Contents

## 0.1 Acknowledgements

The handout is one of the handouts of Gaussian Curvature by the GC Members. This handout is intended for the the learning the inequalities from very basics and it has been authored by Aritra12 , phoenixfire & Rama1728. There will be a next part of the handout which will include jensen , muirhead etcetra. We are also thankful to the several users on AoPS who posted problems and solution and many resource which we noted down or also we forgot some to refer. No handout can be all perfect so if you see any problem or typos don't forget to gmail at gaussiancurvature360@gmail.com

# Chapter 1

# Inequalities: A brief Introduction

## 1.1 Introduction

In this handout, We will be introducing you to some definitions such as homogenity, cyclicity, etc... and some inequalities such as AM-GM-HM, Cauchy Schwartz, etc... We will catch up with some elementary examples followed by some practice problems to ensure that the reader fully understands the concept of inequalities. This is intended for readers who are quite experienced with inequalities and most importantly, have the love and interest for inequalities.

## 1.2 Theory and Examples

In this section, we will be covering some important definitions and some of the important aspects in inequalities, such as AM-GM Inequality, Cauchy - Schwartz Inequality, Holder's Inequality and many more. Nevertheless, every inequality will consist at least 4 to 5 examples to be followed with.

### 1.2.1 Definitions

In this subsection, we give you some important definitions which we will be using in this handout.

**Definition** (Cyclic)— We say that an expression $f(a_1,a_2,...,a_n)$ is cyclic, if for any circular arrangement $a_{i_1},a_{i_2},...,a_{i_n}$ of $a_1,a_2,...,a_n$, we have

$$f(a_1,a_2,...,a_n) = f(a_{i_1},a_{i_2},...,a_{i_n})$$

and the above equality fails for any non-circular arrangement of $a_1,a_2,...,a_n$. For example:

$$f(x,y,z) = x(y-z)+y(z-x)+z(x-y)$$

is cyclic.

A cyclic equality is an equality $f(x,y,z) = 0$ such that you can substitute $(a,b,c) = (g(x,y),g(y,z),g(z,x))$ for some $g(r,s)$ into the equation and get a symmetric equation.

In case of cyclic inequalities can we assume $a = \max(a,b,c)$ without loss of generality. Since all "consecutive" variable pairs $(x,y),(y,z)$ and $(z,x)$ are used in the inequality, the order of the variables matters, but the "rotation" doesn't and that's why you can set $a = \max(a,b,c)$.

**Definition** (Symmetric)— We say that an expression $f(a_1,a_2,...,a_n)$ is symmetric if for any permutation $i_1,i_2,...,i_n$ of $1,2,3,...,n$, we have

$$f(a_1,a_2,...,a_n) = f(a_{i_1},a_{i_2},...,a_{i_n})$$

For example,

$$f(x,y,z) = xy+yz+zx$$

is a symmetric function.

**Definition** (Homogeneous)— We say that an inequality is homogeneous, if all the terms in the inequality have same degree. An inequality involving (multidimensional) polynomials is said to be homogeneous if all the polynomials have the same degree. For example

$$1+\frac{x}{y}+\frac{y}{x} \geq 3$$

is a homogeneous inequality.

**Remark.** This concept of homogeneity is often used in inequalities so that one can "scale" the terms (this is possible because $f(ta_1,ta_2,...,ta_n) = t^k f(a_1,a_2,...,a_n)$ for some fixed $k$), and assume things like the sum of the involved variables is $1$, for things like Jensen's Inequality.

The concept of homogenizing is based on rewriting the number $1$ (in most cases, but any other number can also be used) in terms of our variables (and the given conditions) in a convenient way. We know which such expressions we can make use out of by looking at the degree that we want to obtain. So, we want the highest degree of an already present term.

**Definition** (Normalization)— We first need to clarify the difference between homogeneous functions and non-homogeneous functions. In this case, a condition between variables $x_1, x_2, \ldots, x_n$ such as $x_1 + x_2 + \ldots + x_n = n$ or $x_1 x_2 \ldots x_n = 1$ is meaningless (because we can divide (or multiply) each variable by arbitrary real numbers but the result of the problem is not affected).

Normalisation refers to transforming variables into constants, such as assuming the sum or product of the variables to be $1$. One can also interpret it as the reverse process of homogenisation.

**Definition** (Increasing and Strictly Increasing Functions)— We say that a function $f$ is increasing on a bound $[a,b]$ if for any $x, y \in [a,b]$ such that $x \le y$ we have $f(x) \le f(y)$. $f$ is strictly increasing if the inequality becomes strict.

**Definition** (Decreasing and Strictly Decreasing Functions)— We say that a function $f$ is decreasing on a bound $[a,b]$ if for any $x, y \in [a.b]$ such that $x \le y$, then we have $f(x) \ge f(y)$. $f$ is strictly increasing if the inequality becomes strict.

**Definition** (Convex and Concave)— We say that a function $f(x)$ defined on the interval $[a,b]$ is convex if there exist two points $x_1, x_2$ in $[a,b]$ and $0 < \lambda < 1$ for which

$$f[\lambda x_1 + (1-\lambda)x_2] \le \lambda f(x_1) + (1-\lambda) f(x_2)$$

The function is concave when the inequality sign reverses.

That brings to the end of the definitions, and now we can start learning some new inequalities and solve problems.

### 1.2.2 Trivial Inequality

The trivial inequality states that if $x$ is a real number, then $x^2 \ge 0$. Well, this seems obvious, but this is the root to many other inequalities. From this, the most important result holds,

$$a_1^2 + a_2^2 + \ldots + a_n^2 \ge 0$$

for $a_1, a_2, \ldots, a_n \in \mathbf{R}$, with equality if and only if $a_1 = a_2 = \ldots = a_n = 0$. This result is called the Sum of Squares Inequality, and the equality case is called the Sum of Squares Identity. In this handout, we will be using "SOS" in the place of "Sum of Squares" for simplicity. Now that we are familiar with SOS Inequality and Identity, we can now move forward to some interesting examples.

**Problem 1.2.1.** Let $a,b,c$ be non-negative real numbers. Prove that

$$a^3+b^3+c^3 \geq 3abc$$

**Solution.** We will make use of the identity that

$$a^3+b^3+c^3-3abc = (a+b+c)(a^2+b^2+c^2-ab-bc-ca)$$

Since $a+b+c \geq 0$, it suffices to prove that $a^2+b^2+c^2-ab-bc-ca \geq 0$. Noting that

$$a^2+b^2+c^2-ab-bc-ca = \frac{1}{2}[(a-b)^2+(b-c)^2+(c-a)^2]$$

the inequality follows due to SOS inequality. Equality holds if and only if $a-b=b-c=c-a=0$ or $a=b=c$.

**Problem 1.2.2.** Is it possible to write $b^2-2bc+c^2+1-(bc-b-c)^2$ as the sum of squares each real?

**Solution.** The answer is no. Because, if the above expression can be written as the sum of squares, then the above expression should be non-negative for all values of $b$ and $c$. But when $b=c=3$, then the above expression values to $-8$ which contradicts the assumption that the above expression can be written as the sum of squares.

**Problem 1.2.3.** Let $a,b,c$ be non-negative real numbers. Prove that

$$a^2+b^2+c^2+2abc+1 \geq 2(ab+bc+ca).$$

**Solution.** We can rewrite the above expression as,

$$\frac{2ab[(a+bc-b-c)^2+(c-1)^2]+c(b-1)^2[(a+b-c)^2+1]}{2ab+c(b-1)^2}$$

which is clearly non-negative. Equality holds if and only if $a=b=c=1$.

**Problem 1.2.4** (2018 Romania TST)**.** Given real numbers $x_1,x_2,\ldots,x_n \geq -1$ and $x_1^3+x_2^3+\ldots+x_n^3=0$, find the best constant $c$ for which the inequality $x_1^2+x_2^2+...+x_n^2 \leq cn$ holds true for all tuples $x_1,x_2,...,x_n$.

**Solution.** The key idea is that for any real $x$ such that $x \geq -1$, $(x-2)^2(x+1) \geq 0$. So, we have $x^2 \leq \frac{4}{3} + \frac{x^3}{3}$, and thus $\sum_{i=1}^{n} x_i^2 \leq \sum_{i=1}^{n} \frac{4}{3} + \frac{x_i^3}{3} = \frac{4n}{3}$, and so we conclude that $c = \frac{4}{3}$. Equality holds if and only if $9|n$ and $\frac{8}{9}$ of them are $-1$ and the remaining are $2$.

**Problem 1.2.5.** Find all $n > 1$ such that

$$x_1^2 + x_2^2 + \ldots + x_n^2 \geq x_n(x_1 + x_2 + \ldots + x_{n-1})$$

for all real numbers $x_1, x_2, \ldots, x_n$.

**Solution.** The above inequality rewrites as,

$$x_1^2 - x_1x_n + x_2^2 - x_2x_n + \ldots + x_{n-1}^2 + x_{n-1}x_n + x_n^2 \geq 0$$

or

$$\left(x_1 - \frac{x_n}{2}\right)^2 + \left(x_2 - \frac{x_n}{2}\right)^2 + \ldots + \left(x_{n-1} + \frac{x_n}{2}\right)^2 + x_n^2 - \frac{n-1}{4}x_n^2 \geq 0$$

or

$$\left(x_1 - \frac{x_n}{2}\right)^2 + \left(x_2 - \frac{x_n}{2}\right)^2 + \ldots + \left(x_{n-1} + \frac{x_n}{2}\right)^2 \geq \frac{n-5}{4}x_n^2$$

Clearly for $n = 2,3,4,5$, the right hand side is non-negative and the right hand side is non-positive. If $n > 5$, then we can choose

$$x_1 = x_2 = \ldots = x_{n-1} = \frac{x_n}{2}, x_n = 1$$

and the inequality does not hold true. Thus, the only values for $n$ are $2,3,4,5$.

**Problem 1.2.6 (sqing).** Let $a, b$ be positive real numbers. Prove that

$$a^2 + b^2 \geq \frac{a^2 + b^2 + 2a^2b^2}{ab+1} \geq 2ab$$

**Solution.** Subtracting $2ab$ on both sides, the inequality rewrites as

$$(a-b)^2 \geq \frac{(a-b)^2}{ab+1} \geq 0$$

which reduces to $ab(a-b)^2 \geq 0$ which is obviously true. Equality holds if and only if $a = b$.

**Problem 1.2.7.** (Mathlinks) Let $a, b$ be real numbers such that

$$ab(a^2 - b^2) = a^2 + b^2 + 1$$

Find the minimum of $a^2 + b^2$

**Solution.** Using polar co-ordinates, i.e.

$$a = r\cos\alpha, b = r\sin\alpha, r > 0, \alpha \in [0, 2\pi)$$

we see that the given condition becomes

$$1 + r^2 = r^4 \sin\alpha\cos\alpha(\cos^2\alpha - \sin^2\alpha) = r^4 . \frac{\sin 4\alpha}{4} \le \frac{r^4}{4}$$

which means that

$$r^4 - 4r^2 - 4 \ge 0 \iff (r^2 - 2)^2 - 8 \ge 0$$
$$\iff \left(r^2 - 2 - 2\sqrt{2}\right)\left(r^2 - 2 + 2\sqrt{2}\right) \ge 0$$

Therefore

$$a^2 + b^2 = r^2 \ge 2\left(1 + \sqrt{2}\right)$$

Equality holds if and only if $r = \sqrt{2(1+\sqrt{2})}$ and $\alpha \in \left\{\frac{\pi}{8}, \frac{5\pi}{8}, \frac{9\pi}{8}, \frac{13\pi}{8}\right\}$

**Problem 1.2.8** (1997 USAMO). Let $a, b, c$ be positive real numbers. Prove that

$$\sum_{cyc} \frac{1}{a^3 + b^3 + abc} \le \frac{1}{abc}$$

**Solution.** We first claim that

$$a^3 + b^3 \ge a^2 b + ab^2$$

This is obviously true, since

$$a^3 + b^3 \ge a^2 b + ab^2 \iff (a+b)(a-b)^2 \ge 0$$

Now, observe that

$$\sum_{cyc} \frac{1}{a^3 + b^3 + abc} \le \sum_{cyc} \frac{1}{a^2 b + ab^2 + abc} = \sum_{cyc} \frac{a}{abc(a+b+c)} = \frac{1}{abc}$$

as desired. Equality holds if and only if $a = b = c$.

### 1.2.3 AM-GM-HM Inequality

**Theorem 1.2.1** (AM-GM-HM Inequality) — For positive reals $a_1, a_2, \ldots, a_n$, we have

$$\frac{a_1 + a_2 + \ldots + a_n}{n} \geq \sqrt[n]{a_1 a_2 \ldots a_n} \geq \frac{n}{\frac{1}{a_1} + \frac{1}{a_2} + \ldots + \frac{1}{a_n}}$$

**Remark:** It should be noted that AM-GM works for non-negative reals while AM-HM requires them to be positive.

The AM-GM-HM inequality creates an inequality connection between the well known Arithmetic, Geometric and Harmonic Means in that order. The AM-GM-HM Inequality, especially the AM-GM Inequality is one of the most important inequalities for mathematical competitions. In this section, we will cover some diverse examples based on the AM-GM-HM Inequality. Now that the definition is clear, we can move on to some examples.

**Problem 1.2.9.** Let $a, b, c$ be positive real numbers. Prove that

$$(a^2b + b^2c + c^2a)(ab^2 + bc^2 + ca^2) \geq 3abc(a^2(b+c) + b^2(c+a) + c^2(a+b) - 3abc)$$

**Solution.** Let $x = a^2b + b^2c + c^2a$, $y = ab^2 + bc^2 + ca^2$ and $z = 3abc$. Then, it suffices to prove that

$$xy \geq z(x + y - z)$$

or

$$(x - z)(y - z) \geq 0$$

Now, by the AM-GM Inequality,

$$a^2b + b^2c + c^2a \geq 3\sqrt[3]{a^2b \times b^2c \times c^2a} = 3abc$$

and

$$ab^2 + bc^2 + ca^2 \geq 3\sqrt[3]{ab^2 \times bc^2 \times ca^2} = 3abc$$

Thus, we have $x \geq z$ and $y \geq z$ and so $(x - z)(y - z) \geq 0$ completing the proof. Equality holds if and only if $a = b = c$.

**Problem 1.2.10.** (2002 Romania Junior TST) If $a, b, c \in (0, 1)$, then prove that

$$\sqrt{abc} + \sqrt{(1-a)(1-b)(1-c)} < 1$$

**Solution.** Since $\sqrt{x} < \sqrt[3]{x}$ when $x \in (0,1)$, we have

$$\sqrt{abc} < \sqrt[3]{abc} \leq \frac{a+b+c}{3}$$

by the AM-GM inequality. Similarly, one can get

$$\sqrt{(1-a)(1-b)(1-c)} < \sqrt[3]{(1-a)(1-b)(1-c)} \leq \frac{(1-a)+(1-b)+(1-c)}{3}$$

Thus, we have

$$\sqrt{abc}+\sqrt{(1-a)(1-b)(1-c)} < \sqrt[3]{abc}+\sqrt[3]{(1-a)(1-b)(1-c)} \leq 1$$

completing the proof.

**Problem 1.2.11.** Prove that the inequality

$$\sqrt{a^2+(1-b)^2}+\sqrt{b^2+(1-c)^2}+\sqrt{c^2+(1-a)^2} \geq \frac{3\sqrt{2}}{2}$$

holds for arbitary reals $a,b,c$.

**Solution.** We first prove that for reals $x$ and $y$, we have

$$\sqrt{\frac{x^2+y^2}{2}} \geq \frac{|x|+|y|}{2}$$

the above inequality rewrites as $|x|^2-2|x||y|+|y|^2 \geq 0$ which is obviously true. Equality holds if and only if $|x|=|y|$. Now, applying this result to the original inequality, we have

$$\sqrt{a^2+(1-b)^2} \geq \frac{|a|+|1-b|}{\sqrt{2}}$$

$$\sqrt{b^2+(1-c)^2} \geq \frac{|b|+|1-c|}{\sqrt{2}}$$

$$\sqrt{c^2+(1-a)^2} \geq \frac{|c|+|1-a|}{\sqrt{2}}$$

Also, note that $|x|+|1-x| \geq 1$. Therefore, if we add the above three inequalities, we get the desired result. Equality holds if and only if $a=b=c=\frac{1}{2}$

**Problem 1.2.12.** Let $a,b,c$ be positive reals. Prove that,

$$\frac{6ab-b^2}{8a^2+b^2} < \sqrt{\frac{a}{b}}$$

**Solution.** Let $u=\sqrt{a}$ and $v=\sqrt{b}$. Then, the original inequality rewrites as

$$v(6u^2v^2-v^4) < u(8u^4+v^4)$$

or

$$6u^2v^3 < 8u^5+uv^4+v^5$$

which is obviously true by the AM-GM Inequality. Note that if equality were to hold, then $8u^5=uv^4=v^5$, giving $u=v$ and $8u=v$ which is impossible.

**Problem 1.2.13.** (2015 Iran TST) Let $a,b,c,d$ be positive real numbers such that $\sum_{cyc}\frac{1}{ab}=1$. Prove that

$$abcd+16 \geq 8\sqrt{(a+c)(\frac{1}{a}+\frac{1}{c})}+8\sqrt{(b+d)(\frac{1}{b}+\frac{1}{d})}$$

**Solution.** We have,

$$abcd+16=\sum_{cyc}ab+16\sum_{cyc}\frac{1}{ab}=\sum_{cyc}\left(ab+\frac{16}{bc}\right)\geq 8\sum_{cyc}\sqrt{\frac{a}{c}}=8\sum_{cyc}\sqrt{\frac{a}{c}+2+\frac{c}{a}}=8\sum_{cyc}\sqrt{(a+c)(\frac{1}{a}+\frac{1}{c})}$$

where the last inequality follows from the AM-GM Inequality.

**Problem 1.2.14.** (2021 Hong Kong TST) Find all real triples $(a,b,c)$ satisfying

$$(2^{2a}+1)(2^{2b}+2)(2^{2c}+8)=2^{a+b+c+5}.$$

**Solution.** Let $x=2^a$, $y=2^b$, $z=2^c$ be positive reals. Then the given rewrites as

$$(x^2+1)(y^2+2)(z^2+8)=32xyz.$$

However observe that

$$x^2+1 \geq 2x$$
$$y^2+2 \geq 2y\sqrt{2}$$
$$z^2+8 \geq 4z\sqrt{2}$$

by the AM-GM Inequality. Multiplying gives

$$(x^2+1)(y^2+2)(z^2+8) \geq 32xyz,$$

with equality if and only if $(x,y,z) = (1, \sqrt{2}, 2\sqrt{2})$.
Thus, the only solution is $(a,b,c) = (0, \frac{1}{2}, \frac{3}{2})$.

beginproblem(APMO 1998) Let $x$, $y$, $z$ be positive real numbers. Prove that

**Problem 1.2.15.**

$$\left(1+\frac{x}{y}\right)\left(1+\frac{y}{z}\right)\left(1+\frac{z}{x}\right) \geq 2\left(1+\frac{x+y+z}{\sqrt[3]{xyz}}\right)$$

.

**Solution.** We begin by observing that,

$$\prod\left(1+\frac{x}{y}\right) = \sum_{cyc}\left(\frac{x}{y}+\frac{y}{x}\right)+2$$

By AM-GM,

$$\frac{x}{x}+\frac{x}{y}+\frac{x}{z} \geq \frac{3x}{\sqrt[3]{xyz}}$$

When we add these cyclic relations, we obtain

$$\frac{3+\sum_{cyc}\left(\frac{x}{y}+\frac{y}{x}\right)}{3} \geq \frac{x+y+z}{\sqrt[3]{xyz}}$$

$$\frac{4+\prod\left(1+\frac{x}{y}\right)}{3} \geq 1+\frac{x+y+z}{\sqrt[3]{xyz}}$$

Let

$$\prod\left(1+\frac{x}{y}\right) = a$$

It suffices to prove that,

$$\frac{a}{2} \geq \frac{4+a}{3}$$

or

$$a \geq 8$$

which is obviously true!

**Problem 1.2.16** (sqing). Let $a,b \in (0,1)$ and $4(a+b) = 4ab+3$. Prove that

$$a+2b \leq 3-\sqrt{2}.$$

**Solution.** By the AM-GM Inequality,

$$a = \frac{3-4b}{4-4b} = 1 - \frac{1}{4(1-b)}$$

$$a+2b = 3 - \frac{1}{4(1-b)} - 2(1-b) \leq 3-\sqrt{2}$$

Equality holds if and only if $(1-a) = 2(1-b)$, i.e. $1-\frac{\sqrt{2}}{2}$ and $b = 1-\frac{\sqrt{2}}{4}$

**Problem 1.2.17.** Let $a,b,c$ be positive real numbers. Prove that

$$a^3+b^3+c^3 \geq 2abc\left(\frac{a^2}{b^2+c^2}+\frac{b^2}{c^2+a^2}+\frac{c^2}{a^2+b^2}\right) \geq 3abc$$

**Solution.** Clearly, we have

$$\frac{a^2}{bc} - 2\frac{a^2}{(b^2+c^2)} = \frac{a^2}{bc(b^2+c^2)}(b-c)^2 \geq 0$$

Thus, we have

$$\sum_{cyc}\frac{a^2}{bc} \geq 2\sum_{cyc}\frac{a^2}{b^2+c^2}$$

or

$$a^3+b^2+c^3 \geq 2abc(\frac{a^2}{b^2+c^2}+\frac{b^2}{c^2+a^2}+\frac{c^2}{a^2+b^2})$$

proving the first part of the inequality. As for the second part, it suffices to prove that

$$\frac{a^2}{b^2+c^2}+\frac{b^2}{c^2+a^2}+\frac{c^2}{a^2+b^2} \geq \frac{3}{2}$$

By the AM-HM Inequality, we have

$$\left[(a^2+b^2)+(b^2+c^2)+(c^2+a^2)\right]\left[\frac{1}{a^2+b^2}+\frac{1}{b^2+c^2}+\frac{1}{c^2+a^2}\right] \geq 9$$

or

$$\frac{a^2}{b^2+c^2}+\frac{b^2}{c^2+a^2}+\frac{c^2}{a^2+b^2} \geq \frac{3}{2}$$

proving the second part also. Equality holds if and only if $a=b=c$.

**Problem 1.2.18.** Prove that for all real positive numbers $a, b, c$,

$$\frac{1}{a+ab}+\frac{1}{b+bc}+\frac{1}{c+ca}\geq\frac{3}{1+abc}$$

**Solution.** We can rewrite the inequality as,

$$\sum_{cyc}\frac{1+abc}{a+ab}\geq 3$$

Adding 1 on each term of the inequality, we get

$$\sum_{cyc}\frac{1+a+ab+abc}{a+ab}\geq 6$$

or

$$\sum_{cyc}\frac{a+1}{b(b+1)}+\sum_{cyc}\frac{b(c+1)}{(b+1)}\geq 6$$

which is obviously true by the AM-GM Inequality.

**Problem 1.2.19** (sqing). Let $a, b, c > 0$ and $ab+bc+ca = 3$. Prove that

$$a\sqrt{4b^2+\frac{5}{b}}+b\sqrt{4c^2+\frac{5}{c}}+c\sqrt{4a^2+\frac{5}{a}}\geq 9$$

**Solution.** By the AM-GM Inequality, $(a+b+c)^2 \geq 3(ab+bc+ca) = 9$ and so $a+b+c \geq 3$. Now, notice that,

$$\sqrt{4b^2+\frac{5}{b}}\geq(\frac{b+5}{2})^2$$

$$\iff 3b^3-2b^2-5b+4\geq 0$$

$$\iff (b-1)^2(3b+4)\geq 0$$

which is obviously true. Thus,

$$a\sqrt{4b^2+\frac{5}{b}}+b\sqrt{4c^2+\frac{5}{c}}+c\sqrt{4a^2+\frac{5}{a}}\geq\frac{1}{2}(ab+bc+ca+5(a+b+c))\geq\frac{1}{2}(3+5\times 3)=9$$

completing the proof. Equality holds if and only if $a = b = c = 1$.

**Problem 1.2.20** (sqing). Let $a, b, c > 0$ and $ab+bc+ca = 3$. Prove that

$$a\sqrt{b^2-b+1}+b\sqrt{c^2-c+1}+c\sqrt{a^2-a+1}\geq 3$$

**Solution.** By the AM-GM Inequality, $(a+b+c)^2 \geq 3(ab+bc+ca) = 9$ giving $a+b+c \geq 3$. Again, by the AM-GM Inequality,

$$a^2 - a + 1 \geq a^2 - \frac{a^2+1}{2} + 1 = \frac{a^2+1}{2} \geq \frac{(a+1)^2}{4}$$

Thus, we have

$$\sum_{cyc} a\sqrt{b^2 - b + 1} \geq \frac{1}{2}(ab + bc + ca + a + b + c) \geq 3$$

completing the proof. Equality holds if and only if $a = b = c = 1$.

**Problem 1.2.21.** (Mathlinks) Let $a, b, c$ be nonnegative real numbers such that $a+b+c = 2$. Prove that

$$3abc + \sqrt{4 + a^2b^2c^2} \geq (a+b)(b+c)(c+a)$$

**Solution.** Without Loss of Generality, we may assume that $c = min\{a, b, c\}$, which means that $2 - 3c \geq 0$. Since

$$(a+b)(b+c)(c+a) \geq 8abc \geq 3abc$$

using the fact that $(a+b)(b+c)(c+a) = (a+b+c)(ab+bc+ca) - abc$ and squaring the inequality, we obtain

$$\sqrt{4 + a^2b^2c^2} \geq 2(ab+bc+ca) - 4abc$$

or

$$4 + a^2b^2c^2 \geq 4(ab+bc+ca)^2 - 16abc(ab+bc+ca) + 16a^2b^2c^2$$

or

$$[2(ab+bc+ca) - 5abc][2(ab+bc+ca) - 3abc] \leq 4$$

But

$$LHS \leq [2(ab+bc+ca) - 3abc]^2$$

and so it is enough to prove that

$$2(ab+bc+ca) - 3abc \leq 2$$

or

$$ab(2-3c) + 2c(2-c) \leq 2$$

By the AM-GM Inequality, we have

$$\sqrt{ab} \leq \frac{a+b}{2} \leq \frac{a+b+c}{2} = 1$$

hence

$$ab(2-3c) \leq \sqrt{ab}(2-3c) \leq \frac{2-c}{2}(2-3c)$$

Therefore, it suffices to prove that

$$\frac{2-c}{2}(2-3c)+2c(2-c) \leq 2 \iff (2-c)(2+c) \leq 4$$

which is obviously true. Equality holds if and only if $(a,b,c)=(1,1,0)$ and it's permutations.

### 1.2.4 Cauchy-Schwartz Inequality

**Theorem 1.2.2** (CS Inequality) — For any two real sequences $\{a_i\}_{i=1}^n$ and $\{b_i\}_{i=1}^n$ we have

$$\left(\sum_{i=1}^n a_i^2\right)\left(\sum_{i=1}^n b_i^2\right) \geq \left(\sum_{i=1}^n a_i b_i\right)^2$$

This inequality is just a direct result from the well known Lagrange Identity, which states that for reals $a_1,a_2,\ldots,a_n$ and $b_1,b_2,\ldots,b_n$, we have

$$\left(\sum_{k=1}^n a_k b_k\right)^2 = \left(\sum_{k=1}^n a_k\right)^2\left(\sum_{k=1}^n b_k\right)^2 - \sum_{1\leq k<j\leq 1}(a_k b_j - a_j b_k)^2$$

since

$$\left(\sum_{cyc} a_k b_k\right)^2 = \left(\sum_{k=1}^n a_k\right)^2\left(\sum_{k=1}^n b_k\right)^2 - \sum_{1\leq k<j\leq 1}(a_k b_j - a_j b_k)^2 \leq \left(\sum_{k=1} a_k b_k\right)^2$$

the conclusion follows. The Cauchy-Schwartz Inequality is more applicable when you are given an inequality problem, whose variables are in the domain of reals, because the AM-GM Inequality is only applicable for positive reals, whereas the Cauchy-Schwartz Inequality need not be restricted in the domain of positive reals. Now that we are clear with the Cauchy-Schwartz Inequality, we now introduce to the reader, "Titu's Lemma" which is a corollary of the Cauchy-Schwartz Inequality.

**Theorem 1.2.3** (Titu's Lemma) — For any two real sequences $\{a_i\}_{i=1}^n$ and $\{b_i\}_{i=1}^n$ we have

$$\sum_{i=1}^n \frac{a_i^2}{b_i} \geq \frac{\left(\sum_{i=1}^n a_i\right)^2}{\sum_{i=1}^n b_i}$$

The proof of this lemma is left as an easy exercise to the reader. Now that we are clear with the statements, we now move forward to encounter diverse problems involving the Cauchy-Schwartz Inequality.

**Problem 1.2.22** (2020 Azerbaijan Math Olympiad). Let $a,b,c$ be positive real numbers such that $a+b+c=3$. Prove that

$$\sum_{cyc}\frac{a^2+6}{2a^2+2b^2+2c^2+2a-1}\leq 3$$

**Solution.** By the Cauchy-Schwartz Inequality, we have

$$2(b^2+c^2)\geq(b+c)^2=(3-a)^2=a^2-6a+9$$

Thus,

$$2a^2+2b^2+2c^2+2a-1\geq 3a^2-4a+8$$

Now, the inequality

$$1\geq\frac{a^2+6}{2a^2+2b^2+2c^2+2a-1}$$

is equivalent to each of the following inequalities,

$$1\geq\frac{a^2+6}{3a^2-4a+8}$$

$$3a^2-4a+8\geq a^2+6$$

$$2(a-1)^2\geq 0$$

which is obviously true. Similarly, we have

$$1\geq\frac{b^2+6}{2a^2+2b^2+2c^2+2b-1}$$

and

$$1\geq\frac{c^2+6}{2a^2+2b^2+2c^2+2c-1}$$

Adding the three inequalities, we get the desired result. Equality holds if and only if $a=b=c=1$.

**Problem 1.2.23** (Rama1728). Let $a,b,c$ be real numbers such that $abc=1$. Prove that

$$\frac{a^3}{a^3+2}+\frac{b^3}{b^3+2}+\frac{c^3}{c^3+2}\geq 1$$

**Solution.** Since $abc = 1$, we can write $bc = \frac{1}{a}$, $ca = \frac{1}{b}$ and $ab = \frac{1}{c}$. By Titu's Lemma,

$$\sum_{cyc} \frac{a^3}{a^3+2} = \sum_{cyc} \frac{a^2}{a^2+\frac{2}{a}} = \sum_{cyc} \frac{a^2}{a^2+2bc} \geq \frac{(a+b+c)^2}{a^2+b^2+c^2+2(ab+bc+ca)} = 1$$

completing the proof. Equality holds if and only if $a = b = c = 1$.

**Problem 1.2.24.** Let $a, b, c, d$ be positive real numbers such that $a+b+c+d = 1$. Prove that

$$\sum_{cyc} \frac{abc}{1+bc} \leq \frac{1}{17}$$

**Solution.** We can rewrite the inequality as

$$\sum_{cyc} \frac{a}{1+bc} \geq \frac{16}{17}$$

Now, by Titu's Lemma,

$$\sum_{cyc} \frac{a}{1+bc} = \sum_{cyc} \frac{a^2}{a+abc} \geq \frac{1}{1+\sum_{cyc} abc}$$

Thus, it suffices to prove that

$$\sum_{cyc} abc \leq \frac{1}{16}$$

or

$$ab(c+d) + cd(a+b) \leq \frac{1}{16}$$

or

$$\frac{(a+b)^2}{4}(c+d) + \frac{(c+d)^2}{4}(a+b) \leq \frac{1}{16}$$

or

$$\left[\frac{(a+b)(c+d)}{4}\right] \leq \frac{1}{16}$$

which is obviously true by the AM-GM Inequality as,

$$(a+b)(c+d) \leq \frac{(a+b+c+d)^2}{4} = \frac{1}{4}$$

completing the proof. Equality holds if and only if $a = b = c = d = \frac{1}{4}$.

**Problem 1.2.25** (2020 Moldova TST). Let $a,b,c$ be positive reals. Prove that

$$\sum_{cyc} \frac{a}{\sqrt{7a^2+b^2+c^2}} \leq 1$$

**Solution.** Multiply the inequality by 3. By the AM-GM Inequality, we have

$$\sum_{cyc} \sqrt{\frac{9a^2}{7a^2+b^2+c^2}.1} \leq \sum \frac{\frac{9a^2}{7a^2+b^2+c^2}+1}{2} \leq 3 \iff \sum \frac{3a^2}{7a^2+b^2+c^2} \leq 1$$

by subtracting the inequality by $\frac{1}{2}$, and multiplying by $-1$, we get

$$\sum \frac{a^2+b^2+c^2}{7a^2+b^2+c^2} \geq 1$$

which is obviously true by Titu's Lemma. Equality holds if and only if $a=b=c$

**Problem 1.2.26** (Vasile Cîrtoaje, 2011 ). If $a,b,c$ are real numbers, then

$$32\left(a^2+bc\right)\left(b^2+ca\right)\left(c^2+ab\right)+9(a-b)^2(b-c)^2(c-a)^2 \geq 0$$

**Solution** [ Vo Quoc Ba Can]. For $a,b,c \geq 0$, the inequality is trivial. Otherwise, since the inequality is symmetric and does not change by substituting $-a,-b,-c$ for $a,b,c$, we may assume that $a \leq 0$ and $b,c \geq 0$. Substituting $-a$ for $a$, we need to prove that

$$32\left(a^2+bc\right)\left(b^2-ac\right)\left(c^2-ab\right)+9(a+b)^2(a+c)^2(b-c)^2 \geq 0$$

for all $a,b,c \geq 0$. By the AM-GM inequality, we have

$$(a+b)^2(a+c)^2=\left[a(b+c)+\left(a^2+bc\right)\right]^2 \geq 4a(b+c)\left(a^2+bc\right)$$

Thus, it suffices to prove that

$$8\left(b^2-ac\right)\left(c^2-ab\right)+9a(b+c)(b-c)^2 \geq 0$$

Since

$$\begin{aligned}\left(b^2-ac\right)\left(c^2-ab\right) &= bc\left(bc+a^2\right)-a\left(b^3+c^3\right)\\ &\geq 2abc\sqrt{bc}-a\left(b^3+c^3\right)=-a(b\sqrt{b}-c\sqrt{c})^2,\end{aligned}$$

it is enough to show that

$$9(b+c)(b-c)^2-8(b\sqrt{b}-c\sqrt{c})^2 \geq 0$$

Setting $\sqrt{b} = x$ and $\sqrt{c} = y$, the inequality can be rewritten as

$$(x-y)^2\left[9\left(x^2+y^2\right)(x+y)^2-8\left(x^2+xy+y^2\right)^2\right]\geq 0$$

This follows from the Cauchy-Schwarz inequality as follows

$$9\left(x^2+y^2\right)(x+y)^2 = 9\left[(x-y)^2+2xy\right]\left[(x-y)^2+4xy\right]$$

$$\geq 9\left[(x-y)^2+2\sqrt{2}xy\right]^2 \geq 9\left[\tfrac{2\sqrt{2}}{3}(x-y)^2+2\sqrt{2}xy\right]^2$$

$$= 8\left(x^2+xy+y^2\right)^2 \geq 0$$

The equality occurs when two of $a,b,c$ are zero, and when $-a=b=c$ (or any cyclic permutation).

**Problem 1.2.27** (Vasile Cîrtoaje). If $a,b$, c are real numbers, then

$$a^2\left(b^2-c^2\right)^2+b^2\left(c^2-a^2\right)^2+c^2\left(a^2-b^2\right)^2 \geq \frac{3}{8}(a-b)^2(b-c)^2(c-a)^2$$

**Solution.** We see that the inequality remains unchanged and the product

$$(a+b)(b+c)(c+a)$$

changes its sign by replacing $a,b,c$ with $-a,-b,-c$, respectively. Thus, without loss of generality, we may assume that $(a+b)(b+c)(c+a)\geq 0$. According to this condition, at least one of $a,b,c$ is non negative. So, we may consider $a \geq 0$, and hence

$$a(a+b)(b+c)(c+a)\geq 0$$

In virtue of the Cauchy-Schwarz inequality, we get

$$b^2\left(c^2-a^2\right)^2+c^2\left(a^2-b^2\right)^2 \geq \frac{1}{2}\left[b\left(c^2-a^2\right)+c\left(a^2-b^2\right)\right]^2 = \frac{1}{2}(b-c)^2\left(a^2+bc\right)^2$$

Thus, it suffices to show the equivalent inequality,

$$\begin{gathered}(a+b)(a+c)\left[a^2+5a(b+c)+bc\right]\geq 0\\ (a+b)(a+c)[(a+b)(a+c)+4a(b+c)]\geq 0\\ (a+b)^2(a+c)^2+4a(a+b)(b+c)(c+a)\geq 0\end{gathered}$$

Since the last inequality is clearly true, the proof is completed. The equality holds for $a=b=c$, for $-a=b=c$ (or any cyclic permutation), and for $b=c=0$ (or any cyclic permutation).

**Problem 1.2.28** (sqing). Let $a,b,c>0$ and $a+b+c=3$. Prove that

$$\frac{a}{2a+bc}+\frac{b}{2b+ca}+\frac{c}{2c+ab}\le 1.$$

**Solution.** The inequality rewrites as,

$$\sum_{cyc}\frac{bc}{2a+bc}\ge 1$$

By Titu's Lemma,

$$\sum_{cyc}\frac{bc}{2a+bc}=\sum_{cyc}\frac{b^2c^2}{2abc+b^2c^2}\ge\frac{(ab+bc+ca)^2}{6abc+a^2b^2+b^2c^2+c^2a^2}=1$$

completing the proof. Equality holds if and only if $a=b=c=1$.

**Problem 1.2.29** (justin 1228). Let $a,b,c,d$ be positive real numbers. Prove that

$$\frac{a}{b+c}+\frac{b}{c+d}+\frac{c}{d+a}+\frac{d}{a+b}\ge 2$$

**Solution.** By Titu's Lemma, we have

$$\sum_{cyc}\frac{a}{b+c}=\sum_{cyc}\frac{a^2}{ab+ac}\ge\frac{(a+b+c+d)^2}{ab+bc+cd+da+2ac+2bd}$$

Thus, it suffices to prove that

$$(a+b+c+d)^2\ge 2ab+2bc+2cd+2da+4ac+4bd$$

or

$$(a-c)^2+(b-d)^2\ge 0$$

which is obviously true by the trivial inequality. Equality holds if and only if $a=b=c=d$.

**Problem 1.2.30** (IMO 1995). Let $a,b,c$ be positive reals such that $abc=1$. Prove that

$$\frac{1}{a^3(b+c)}+\frac{1}{b^3(c+a)}+\frac{1}{c^3(a+b)}\ge\frac{3}{2}$$

**Solution.** By the Cauchy-Schwarz Inequality, we have

$$\left(\frac{1}{a^3(b+c)}+\frac{1}{b^3(c+a)}+\frac{1}{c^3(a+b)}\right)(a(b+c)+b(c+a)+c(a+b)) \geq \left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)^2$$

since $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=ab+bc+ca$, we have

$$\frac{1}{a^3(b+c)}+\frac{1}{b^3(c+a)}+\frac{1}{c^3(a+b)} \geq \frac{ab+bc+ca}{2} \geq \frac{3\sqrt[3]{a^2b^2c^2}}{2}=\frac{3}{2}$$

as desired. Equality holds if and only if $a=b=c=1$.

Another solution with Titu's Inequality,

**Solution.** After the setting $a=\frac{1}{x}, b=\frac{1}{y}, c=\frac{1}{z}$, and as $abc=1 \iff \frac{1}{abc}=1 \implies xyz=1$.

$$\boxed{\textbf{Claim}: \frac{x^2}{y+z}+\frac{y^2}{z+x}+\frac{z^2}{x+y} \geq \frac{3}{2}}$$

By Titu Lemma,

$$\implies \frac{x^2}{y+z}+\frac{y^2}{z+x}+\frac{z^2}{x+y} \geq \frac{(x+y+z)^2}{2(x+y+z)}$$

$$\implies \frac{x^2}{y+z}+\frac{y^2}{z+x}+\frac{z^2}{x+y} \geq \frac{(x+y+z)}{2}$$

Now by AM-GM we know that

$$(x+y+z) \geq 3\sqrt[3]{xyz}$$

and $xyz=1$ which concludes to $\implies (x+y+z) \geq 3\sqrt[3]{1}$
Therefore we get

$$\implies \frac{x^2}{y+z}+\frac{y^2}{z+x}+\frac{z^2}{x+y} \geq \frac{3}{2}$$

Hence our claim is proved and this proves the original inequality.

**Problem 1.2.31** (Vasile Cîrtoaje). Let $x, y, z$ be positive real numbers. Prove that

$$\frac{x}{y+z}+\frac{y}{z+x}+\frac{z}{x+y} \geq \frac{3}{2}+\frac{27}{16}.\frac{(y-z)^2}{(x+y+z)^2}$$

**Solution.** First suppose that $2x \leq y+z$. Adding the equality

$$\frac{y+z}{x+z}+\frac{x+z}{y+z}-2=\frac{(y-z)^2}{(x+z)(y+z)}$$

and it's symmetric analogs, we see that the inequality can be rewritten as

$$\sum_{cyc}\frac{(x-y)^2}{(x+z)(y+z)} \geq \frac{27}{8}.\frac{(y-z)^2}{(x+y+z)^2}$$

But by Titu's Lemma, we have

$$\frac{(x-y)^2}{(x+z)(y+z)}+\frac{(x-z)^2}{(x+y)(y+z)} \geq \frac{(y-x+x-z)^2}{(x+z)(y+z)+(y+z)(x+y)}=\frac{(y-z)^2}{(y+z)(2x+y+z)}$$

Therefore, it suffices to prove that if $2x \leq y+z$, then

$$\frac{1}{2x+y+z}+\frac{1}{(x+y)(x+z)} \geq \frac{27}{8(x+y+z)^2}$$

With the substitutions $a=x+y, b=y+z, c=z+x$, we must prove that if $2c \geq a+b$, then

$$2(a+b+c)^2(ab+bc+ca) \geq 27abc(a+b)$$

or

$$\left(2+\frac{2c}{a+b}\right)^2\left(1+\frac{c}{a}+\frac{c}{b}\right) \geq 27.\frac{2c}{a+b}$$

By the Cauchy-Schwarz Inequality, we have

$$\frac{c}{a}+\frac{c}{b} \geq \frac{4c}{a+b}$$

Letting $t=\frac{2c}{a+b} \geq 1$, it remains to show that

$$(t+2)^2(1+2t) \geq 27t$$

which is obviously true from the AM-GM Inequality, i.e.

$$(t+2)^2 \geq 9\sqrt[3]{t^2}$$

and

$$1+2t \geq 3\sqrt[3]{t}$$

Now suppose that $2x \geq y+z$. With the same notations as above, $a=x+y, b=y+z, c=z+x$, we have $t=\frac{2c}{a+b} \leq 1$ and the inequality of statement can be rewritten as

$$\frac{a+b}{2c}+\frac{(a+b+c)(a+b)}{2ab} \geq 4+\frac{27}{4}.\frac{(a-b)^2}{(a+b+c)^2}$$

or

$$\frac{a+b}{2c}+\frac{(a+b+c)(a+b)}{2ab}+\frac{27ab}{(a+b+c)^2}\geq 4+\frac{27}{4}.\frac{(a-b)^2}{(a+b+c)^2}$$

But from the AM-GM Inequality, we have

$$\frac{(a+b+c)(a+b)}{2ab}+\frac{27ab}{(a+b+c)}\geq 6\sqrt{\frac{3(a+b)}{2(a+b+c)}}$$

Therefore, it suffices to show that

$$\frac{a+b}{2c}+6\sqrt{\frac{3(a+b)}{2(a+b+c)}}\geq 4+\frac{27}{4}.\frac{(a+b)^2}{(a+b+c)^2}$$

or

$$\frac{1}{t}+6\sqrt{\frac{3}{2+t}}\geq 4+\frac{27}{(2+t)^2}$$

Letting $\frac{3}{2}\geq u^2=\frac{3}{2+t}\geq 1$, this inequality becomes

$$\frac{u^2}{3-2u^2}+6u\geq 4+3u^4 \iff (u-1)^2[6(u^3-1)(u+2)+9u^2]\geq 0$$

which is obviously true.

**Problem 1.2.32.** Let $a_1,a_2,...,a_n$ be positive numbers such that $a_1a_2...a_n=1$. Prove

$$\frac{1}{1+(n-1)a_1}+\frac{1}{1+(n-1)a_2}+\cdot+\frac{1}{1+(n-1)a_n}\geq 1$$

**Solution.** Using the substitution $a_i=\frac{1}{x_i}$ for all $i$, the inequality becomes

$$\frac{x_1}{x_1+n-1}+\frac{x_2}{x_2+n-1}+\cdots+\frac{x_n}{x_n+n-1}\geq 1$$

where $x_1,x_2,\ldots,x_n$ are positive numbers such that $x_1x_2\ldots x_n=1$. By the Cauchy-Schwarz Inequality, we have

$$\sum\frac{x_1}{x_1+n-1}\geq\frac{\left(\sqrt{x_1}+\sqrt{x_2}+\cdots+\sqrt{x_n}\right)^2}{\sum(x_1+n-1)}$$

Thus, we still have to show

$$(\sqrt{x_1}+\sqrt{x_2}+\cdots+\sqrt{x_n})^2\geq n(n-1)+\sum x_1$$

which is equivalent to

$$\sum_{1\le i<j\le n} \sqrt{x_i x_j} \ge \frac{n(n-1)}{2}$$

Since $x_1 x_2 \quad x_n = 1$, the inequality follows immediately from the AM-GM Inequality

**Problem 1.2.33.** Let $a,b,c > 0$ and $a+b+c = 3$. Prove that

$$\frac{1}{3+a^2+b^2} + \frac{1}{3+b^2+c^2} + \frac{1}{3+c^2+a^2} \le \frac{3}{5}$$

**Solution.** We have:

$$\frac{1}{3+a^2+b^2} + \frac{1}{3+b^2+c^2} + \frac{1}{3+c^2+a^2} \le \frac{3}{5}$$

$$\iff \frac{3}{3+a^2+b^2} + \frac{3}{3+b^2+c^2} + \frac{3}{3+c^2+a^2} \le \frac{9}{5}$$

$$\sum \frac{a^2+b^2}{3+a^2+b^2} \ge \frac{6}{5}$$

Using Cauchy-Schwarz's inequality;

$$\left(\sum \frac{a^2+b^2}{3+a^2+b^2}\right)\left(\sum 3+a^2+b^2\right) \ge \left(\sum \sqrt{a^2+b^2}\right)^2$$

That means We have to prove

$$\left(\sum \sqrt{a^2+b^2}\right)^2 \ge \frac{6}{5}\left(\sum \left(3+a^2+b^2\right)\right)$$

$$\sum \left(a^2+b^2\right) + 2\sum \sqrt{\left(a^2+b^2\right)\left(a^2+c^2\right)} \ge \frac{54}{5} + \frac{12}{5}\sum a^2$$

$$8\sum a^2 + 10\sum ab \ge 54 \iff 5(a+b+c)^2 + 3\sum a^2 \ge 54$$

it is true with $a+b+c = 3$

**Problem 1.2.34** (Mathlinks). Let $a,b,c,d$ be positive real numbers such that

$$a+b+c+d = 4$$

Prove that

$$27\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right) \geq 9\left(a^3+b^3+c^3+d^3\right)+8$$

**Solution.** Assume WLOG that $a \geq b \geq c \geq d$. Note that the equality holds when $a=3, b=3, c=3, d=\frac{1}{3}$. The inequality can be written sucessively in the following forms:

$$\sum_{b,c,d} 9\left[\frac{3}{b}-b^3-9+\frac{1}{27}+\left(27+\frac{1}{3}\right)b-\frac{1}{3}\left(27+\frac{1}{3}\right)\right]$$

$$\geq 9\left[\frac{3}{a}-a^3-9+\frac{1}{27}+\left(27+\frac{1}{3}\right)a-\frac{1}{3}\left(27+\frac{1}{3}\right)\right]$$

or

$$\sum_{b,c,d} \frac{(3b-1)^2(-3b^2-2b+81)}{3b} \geq \frac{3(a-3)^2(3a^2+18a-1)}{a}$$

But $-3x^2-2x+8a \geq -3a^2-a+81$ for $x \in b,c,d$, therefore it suffices to prove that

$$(-3a^2-2a+81)\sum_{b,c,d} \frac{(3b-1)^2}{3b} \geq \frac{3(a-3)^2(3a^2+18a-1)}{a}$$

By the Cauchy-Schwarz Inequality, we have

$$\sum_{b,c,d} \frac{(3b-1)^2}{3b} \geq \frac{[3(b+c+d)-3]^2}{3(b+c+d)} = \frac{3(3-a)^2}{(4-a)}$$

So, it remains to show that

$$\frac{3(3-a)^2(-3a^2-2a-81)}{4-a} \geq \frac{3(a-3)^2(3a^2+18a-1)}{a}$$

or

$$\frac{12(a-3)^2(a+1)^2}{a(4-a)} \geq 0$$

which is obviously true.

### 1.2.5 Holders Inequality

**Theorem 1.2.4** (Holder's Inequality) — Let $\{a_i\}_{i=1}^n$ and $\{b_i\}_{i=1}^n$ be two sequences of positive real numbers, and let $p, q$ be two positive real numbers such that $\frac{1}{p}+\frac{1}{q}=1$. Then, we have

$$\sum_{k=1}^{n} a_k b_k \leq \left(\sum_{k=1}^{n} a_k^p\right)^{\frac{1}{p}} \left(\sum_{k=1}^{n} b_k^q\right)^{\frac{1}{q}}$$

Equality holds if and only if $\frac{a_1^p}{b_1^q}=\frac{a_2^p}{b_2^q}=\cdots=\frac{a_n^p}{b_n^q}$

**Remark:** If $p=q=2$ this becomes Cauchy's inequality. We now give a generalization of Holder's Inequality.

**Theorem 1.2.5** (Generalized Holder's Inequality) — Let $a_{ij1\leq i<j\leq n}$ be a sequence of positive real numbers, and $p_1, p_2, \ldots, p_n$ be positive real numbers such that $p_1+p_2+\ldots+p_n=1$. Then,

$$\sum_{i=1}^{m}\left(\prod_{j=1}^{n} a_{ij}^{p_j}\right) \leq \prod_{j=1}^{n}\left(\sum_{i=1}^{m} a_{ij}\right)^{p_j}$$

**Remark.** The Holder's Inequality is a generalization of the Cauchy-Schwartz Inequality. It is really helpful in proving inequalities which are cyclic and involve product of two cyclic terms.

Now that the statement is clear, we can move on to some problems.

**Problem 1.2.35.** Let $x, y, z > 0$ be real numbers. Prove that

$$(x^2+xy+y^2)(y^2+yz+z^2)(z^2+zx+x^2) \geq (xy+yz+zx)^3$$

**Solution.** Using the Generalized Holder's Inequality, we have

$$\begin{aligned}
&(x^2+xy+y^2)(y^2+yz+z^2)(z^2+zx+x^2)\\
&=(xy+x^2+y^2)(y^2+z^2+yz)(x^2+xz+z^2)\\
&\geq\left(\sqrt[3]{x^2.y^2.xy}+\sqrt[3]{y^2.z^2.yz}+\sqrt[3]{x^2.z^2.xz}\right)^3\\
&=(xy+yz+zx)^3
\end{aligned}$$

**Problem 1.2.36** (Pham Kim Hung). Let $a,b,c$ be positive real numbers. Prove that

$$a^2(b+c)+b^2(c+a)+c^2(a+b) \geq (ab+bc+ca)\sqrt[3]{(a+b)(b+c)(c+a)}$$

**Solution.** Notice that the following expressions are equal to each other

$$\begin{gathered} a^2(b+c)+b^2(c+a)+c^2(a+b) \\ b^2(c+a)+c^2(a+b)+a^2(b+c) \\ ab(a+b)+bc(b+c)+ca(c+a) \end{gathered}$$

According to Hölder inequality, we get that

$$\left(\sum_{cyc} a^2(b+c)\right)^3 \geq \left(\sum_{cyc} ab\sqrt[3]{(a+b)(b+c)(c+a)}\right)^3$$

which is exactly the desired result. Equality holds for $a=b=c$

**Problem 1.2.37** (Pham kim Hung). Let $a,b,c$ be positive real numbers such that $abc=1$. Prove that

$$\frac{a}{\sqrt{7+b+c}}+\frac{b}{\sqrt{7+c+a}}+\frac{c}{\sqrt{7+a+b}} \geq 1$$

$$\frac{a}{\sqrt{7+b^2+c^2}}+\frac{b}{\sqrt{7+c^2+a^2}}+\frac{c}{\sqrt{7+a^2+b^2}} \geq 1$$

With the same condition, determine if the following inequality is true or false.

$$\frac{a}{\sqrt{7+b^3+c^3}}+\frac{b}{\sqrt{7+c^3+a^3}}+\frac{c}{\sqrt{7+a^3+b^3}} \geq 1$$

**Solution.** For the first one, apply Hölder inequality in the following form

$$\left(\sum_{cyc} \frac{a}{\sqrt{7+b+c}}\right)\left(\sum_{cyc} \frac{a}{\sqrt{7+b+c}}\right)\left(\sum_{cyc} a(7+b+c)\right) \geq (a+b+c)^3$$

It's enough to prove that

$$(a+b+c)^3 \geq 7(a+b+c)+2(ab+bc+ca)$$

Because $a+b+c \geq 3\sqrt[3]{abc}=3$

$$(a+b+c)^3 \geq 7(a+b+c)+\frac{2}{3}(a+b+c)^2 \geq 7(a+b+c)+2(ab+bc+ca)$$

For the second one, apply Hölder inequality in the following form

$$\left(\sum_{cyc}\frac{a}{\sqrt{7+b^2+c^2}}\right)\left(\sum_{cyc}\frac{a}{\sqrt{7+b^2+c^2}}\right)\left(\sum_{cyc}a\left(7+b^2+c^2\right)\right)\geq(a+b+c)^3$$

On the other hand

$$\begin{aligned}\sum_{cyc}a\left(7+b^2+c^2\right)=&7(a+b+c)+(a+b+c)(ab+bc+ca)-3abc\\ &\leq 7(a+b+c)+\frac{1}{3}(a+b+c)^3-3\leq(a+b+c)^3\end{aligned}$$

Equality holds for $a=b=c=1$ for both parts. The third one is not true. Indeed, we only need to choose $a\to 0$ and $b=c\to+\infty$, or namely, $a=10^{-4}, b=c=100$.

**Problem 1.2.38** (Titu Andreescu, 2006). If $a,b,c$ are real numbers, then

$$3\left(a^2-ab+b^2\right)\left(b^2-bc+c^2\right)\left(c^2-ca+a^2\right)\geq a^3b^3+b^3c^3+c^3a^3$$

**Solution.** Substituting $a,b,c$ by $|a|,|b|,|c|$, respectively, the left side of the inequality remains unchanged or decreases, while the right side remains unchanged or increases. Therefore, it suffices to prove the inequality for $a,b,c\geq 0$. If $a=0$, then the inequality reduces to $b^2c^2(b-c)^2\geq 0$. Consider further then $a,b,c>0$. We first show that

$$3\left(a^2-ab+b^2\right)^3\geq a^6+a^3b^3+b^6$$

Indeed, setting $x=\frac{a}{b}+\frac{b}{a}, x\geq 2$, we can write this inequality as

$$\begin{aligned}3(x-1)^3&\geq x^3-3x+1\\ \Longleftrightarrow\ (x-2)^2(2x-1)&\geq 0\end{aligned}$$

Using this inequality, together with the similar ones, we have

$$\begin{aligned}&27\left(a^2-ab+b^2\right)^3\left(b^2-bc+c^2\right)^3\left(c^2-ca+a^2\right)^3\\ &\geq\left(a^6+a^3b^3+b^6\right)\left(b^6+b^3c^3+c^6\right)\left(c^6+c^3a^3+a^6\right)\end{aligned}$$

Therefore, it suffices to show that

$$\left(a^6+a^3b^3+b^6\right)\left(b^6+b^3c^3+c^6\right)\left(c^6+c^3a^3+a^6\right)\geq\left(a^3b^3+b^3c^3+c^3a^3\right)^3$$

Writing this inequality in the form

$$\left(a^3b^3+b^6+a^6\right)\left(b^6+b^3c^3+c^6\right)\left(a^6+c^6+c^3a^3\right)\geq\left(a^3b^3+b^3c^3+c^3a^3\right)^3$$

we see that it is just Hölder's Inequality. The equality holds when $a=b=c$, when $a=0$ and $b=c$ (or any cyclic permutation), and when two of $a,b,c$ are 0 .

**Problem 1.2.39** (Titu Andreescu). Let $a,b,c$ be positive real numbers such that $ab+bc+ca=3$. Prove that

$$(2+3a^3)(2+3b^3)(2+3c^3)\geq 125$$

**Solution.** First, by Holder's Inequality,

$$(a^3+1)(1+b^3)(1+1)\geq (a+b)^3$$

$$(b^3+1)(1+c^3)(1+1)\geq (b+c)^3$$

$$(c^3+1)(1+a^3)(1+1)\geq (c+a)^3$$

Multiplying the inequalities, we get

$$(a^3+1)(b^3+1)(c^3+1)\geq \sqrt{\left[\frac{(a+b)(b+c)(c+a)}{2}\right]^3}$$

But, by the AM-GM Inequality and the well known inequality $(a+b+c)^2\geq 3(ab+bc+ca)$, we get

$$(a+b)(b+c)(c+a)=(a+b+c)(ab+bc+ca)-abc$$

$$\geq (a+b+c)(ab+bc+ca)-\frac{(a+b+c)(ab+bc+ca)}{9}=\frac{8(a+b+c)}{3}\geq\frac{8\sqrt{3(ab+bc+ca)}}{3}=8$$

Therefore

$$(a^3+1)(b^3+1)(c^3+1)\geq 8$$

Returning to the initial problem, applying Holder's Inequality again, we have

$$\left[(1+a^3)+a^3+a^3+1\right]\left[(1+b^3)+b^3+b^3+1\right]\left[(1+c^3)+c^3+c^3+1\right]$$

$$\geq\left[\sqrt[3]{(1+a^3)(1+b^3)(1+c^3)}+ab+bc+ca\right]^3$$

$$\geq (2+3)^3=125$$

Equality holds if and only if $a=b=c=1$.

**Problem 1.2.40.** Suppose that $a,b,c$ are positive real numbers satisfying the condition $3\max\left(a^2,b^2,c^2\right)\leq$

$2\left(a^2+b^2+c^2\right)$. Prove that

$$\frac{a}{\sqrt{2b^2+2c^2-a^2}}+\frac{b}{\sqrt{2c^2+2a^2-b^2}}+\frac{c}{\sqrt{2a^2+2b^2-c^2}}\geq\sqrt{3}$$

**Solution.** By Holder, we deduce that

$$\left(\sum_{cyc}\frac{a}{\sqrt{2b^2+2c^2-a^2}}\right)\left(\sum_{cyc}\frac{a}{\sqrt{2b^2+2c^2-a^2}}\right)\left(\sum_{cyc}a\left(2b^2+2c^2-a^2\right)\right)\geq(a+b+c)^3$$

It remains to prove that

$$(a+b+c)^3\geq 3\sum_{cyc}a\left(2b^2+2c^2-a^2\right)$$

Rewrite this one in the following form

$$3\left(abc-\prod_{cyc}(a-b+c)\right)+2\left(\sum_{cyc}a^3-3abc\right)\geq 0$$

which is obvious (for a quick proof that the first term is bigger than $0$, replace $a-b+c=x$. etc). Equality holds for $a=b=c$

**Problem 1.2.41** (IMO SL 2018). Find the maximal value of

$$S=\sqrt[3]{\frac{a}{b+7}}+\sqrt[3]{\frac{b}{c+7}}+\sqrt[3]{\frac{c}{d+7}}+\sqrt[3]{\frac{d}{a+7}},$$

where $a$, $b$, $c$, $d$ are nonnegative real numbers which satisfy $a+b+c+d=100$.

**Solution** [Evan Chen]. The answer is $\frac{8}{\sqrt[3]{7}}$ achieved when $(a,b,c,d)=(49,1,49,1)$. Let $S$ be desired maximum. Set $a=w^2$, $b=x^2$, $c=y^2$, $d=z^2$ and moreover define $t=w+x+y+z$. First, by Hölder inequality we have

$$S^3\leq\left(\sum_{\text{cyc}}w\right)\left(\sum_{\text{cyc}}w\right)\left(\sum_{\text{cyc}}\frac{1}{x^2+7}\right).$$

Also, we use will use the key estimate

$$\begin{aligned} 0 &\le \frac{(x-1)^2(x-7)^2}{x^2+7} \\ &= x^2-16x+71-\frac{448}{x^2+7} \\ \implies \frac{1}{x^2+7} &\le \frac{x^2-16x+71}{448} \\ \implies \sum_{\text{cyc}} \frac{1}{x^2+7} &\le \sum_{\text{cyc}} \frac{x^2-16x+71}{448} = \frac{6}{7}-\frac{t}{28} \end{aligned}$$

Consequently, the earlier estimate gives

$$S^3 \le \frac{t^2(24-t)}{28} = \frac{t\cdot t\cdot(48-2t)}{56} \le \frac{16^3}{56} = \frac{512}{7}$$

as desired, with the last inequality by AM-GM

### 1.2.6 Aczel's Inequality

**Theorem 1.2.6** (Aczel's Inequality) — For all real numbers $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ such that $a_1^2 > a_2^2 + a_3^2 + \ldots + a_n^2$ and $b_1^2 > b_2^2 + b_3^2 + \ldots + b_n^2$, we have

$$(a_1b_1 - a_2b_2 - \ldots - a_nb_n)^2 \ge (a_1^2 - a_2^2 - \ldots - a_n^2)(b_1^2 - b_2^2 \ldots - b_n^2)$$

Equality holds if and only if $b_i = ta_i$ for some real $t$ or $a_i = b_i = 0$ for $i = 1, 2, 3, \ldots, n$.

**Remark.** Acel's Inequality is similar to the Cauchy Schwarz Inequality, and both their proofs are similar by the use of discriminant. We invite the courageous reader to prove Aczel's Inequality.

**Problem 1.2.42** (Titu Andreescu). In a triangle $ABC$, $\angle C > 90^\circ$ and

$$3a + \sqrt{15ab} + 5b = 7c$$

where $a, b, c$ are the sides of $ABC$. Prove that $\angle C \le 120^\circ$

**Solution.** Since $c^2 - a^2 - b^2 > 0$ and $7^2 - 3^2 - 5^2 > 0$, by Aczel's Inequality, we have

$$15ab = (7c - 3a - 5b)^2 \geq (7^2 - 3^2 - 5^2)(c^2 - a^2 - b^2) = 15(-2ab\cos C)$$

giving $\cos C \leq -\frac{1}{2}$ and thus $\angle C \leq 120$, completing the proof. Equality holds if and only if $\frac{a}{3} = \frac{b}{5} = \frac{c}{7}$

**Problem 1.2.43** (USA TST 2004). Suppose $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ are real numbers such that

$$(a_1^2 + a_2^2 + \cdots + a_n^2 - 1)(b_1^2 + b_2^2 + \cdots + b_n^2 - 1) > (a_1b_1 + a_2b_2 + \cdots + a_nb_n - 1)^2.$$

Prove that $a_1^2 + a_2^2 + \cdots + a_n^2 > 1$ and $b_1^2 + b_2^2 + \cdots + b_n^2 > 1$.

**Solution.** Assume the contrary, that $1 > a_1^2 + a_2^2 + \ldots + a_n^2$ and $1 > b_1^2 + b_2^2 + \ldots + b_n^2$. Then, by Aczel's Inequality, we have

$$(1 - a_1b_1 - a_2b_2 - \ldots - a_nb_n) \geq (1 - a_1^2 - a_2^2 - \ldots - a_n^2)(1 - b_1^2 - b_2^2 - \ldots - b_n^2)$$

contrary to the hypothesis.

**Problem 1.2.44** (Rama1728). Let $ABCD$ be a quadrilateral with $\angle BAD > 90^\circ$ and $\angle ACD > 90^\circ$. If $AD = a, AB = b, BD = c, AC = 3, CD = 5$, prove that

$$34c^2 + a^4 + (ab + 15)^2 \geq 2ac(3a + 5b) + (5a)^2 + (3b)^2 + (15)^2$$

**Solution.** The inequality rewrites as

$$(ca - 3a - 5b)^2 \geq (c^2 - a^2 - b^2)(a^2 - 3^2 - 5^2)$$

which follows from Aczel's Inequality, as $c^2 > a^2 + b^2$ and $a^2 > 3^2 + 5^2$. The last two inequalities follow from the fact that triangles $ABD$ and $ACD$ are obtuse. Equality holds if and only if $\frac{a}{c} = \frac{3}{a} = \frac{5}{b}$

**Problem 1.2.45** (Shanghai JHSMC 2017). Let $x, y$ be real numbers such that

$$\sqrt{1 - \frac{x^2}{4}} + \sqrt{1 - \frac{y^2}{16}} = \frac{3}{2}$$

Find the maximum value of $xy$.

**Solution.** By Aczel's Inequality, we have

$$\left(1-\frac{x^2}{4}\right)\left(1-\frac{y^2}{16}\right) \leq \left(1-\frac{xy}{8}\right)^2$$

By squaring given condition and applying the above result, we get

$$\frac{9}{4} = 1-\frac{x^2}{4}+1-\frac{y^2}{16}+2\sqrt{\left(1-\frac{x^2}{4}\right)\left(1-\frac{y^2}{16}\right)}$$

$$\leq 2-\frac{x^2}{4}-\frac{y^2}{16}+2\left(1-\frac{xy}{8}\right)$$

$$\leq 2-\frac{xy}{4}+2-\frac{xy}{4} = 4-\frac{xy}{2}$$

Thus, the maximum of $xy$ is $\frac{7}{2}$ with equality if and only if $y = 2x = \sqrt{7}$

**Problem 1.2.46** (Titu Andreescu). Let $a,b,c,d,x,y,z,t$ be positive real numbers such that $a^2+b^2+c^2+d^2 = x^2+y^2+z^2+t^2 = 1$. Prove that

$$ax+\sqrt{(b^2+c^2)(y^2+z^2)}+dt \leq 1$$

**Solution.** We have $1-a^2-d^2 > 0$ and $1-x^2-t^2 > 0$. Thus, by Aczel's Inequality, we have

$$(1-ax-dt)^2 \geq (1-a^2-d^2)(1-x^2-t^2) = (b^2+c^2)(y^2+z^2)$$

By the Cauchy-Schwartz Inequality, we have

$$ax+dt \leq \sqrt{(a^2+d^2)(x^2+t^2)} < 1$$

Therefore,

$$1-ax-dt \geq \sqrt{(b^2+c^2)(y^2+z^2)}$$

hence the conclusion. Equality holds if and only if $a = x, d = t$

## 1.2.7 Rearrangement Inequality

**Theorem 1.2.7** (Rearrangement Inequality) — Let $a_1 \leq a_2 \leq a_3 .... \leq a_n$ and $b_1 \leq b_2 \leq b_3 ... \leq b_n$ be real numbers. For any permutation $c_1, c_2, ..., c_n$ of $b_1, b_2, ...b_n$, we have

$$\sum_{i=1}^{n} a_{n-i+1} b_i \leq \sum_{i=1}^{n} a_i c_i \leq \sum_{i=1}^{n} a_i b_i$$

**Problem 1.2.47** (Nesbitt's Inequality). Let $a, b, c$ be positive real numbers. Prove that

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{3}{2}$$

**Solution.** Note that $a, b, c$ and $\frac{1}{b+c} = \frac{1}{a+b+c-a}$, $\frac{1}{c+a} = \frac{1}{a+b+c-b}$, $\frac{1}{a+b} = \frac{1}{a+b+c-c}$ are sorted in the same order. Then by the rearrangement inequality,

$$2\left(\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b}\right) \geq \frac{b}{b+c} + \frac{c}{b+c} + \frac{c}{c+a} + \frac{a}{c+a} + \frac{a}{a+b} + \frac{b}{a+b} = 3$$

.
For equality to occur, since we changed $a \cdot \frac{1}{b+c} + b \cdot \frac{1}{c+a}$ to $b \cdot \frac{1}{b+c} + a \cdot \frac{1}{c+a}$, we must have $a = b$, so by symmetry, all the variables must be equal.

**Problem 1.2.48.** Let $a, b, c$ be nonnegative reals such that $a + b + c = 1$. Prove that

$$a^3 b + b^3 c + c^3 a + 3abc \leq \frac{4}{27}$$

**Solution.** Let $\{a, b, c\} = \{x, y, z\}$, where $x \geq y \geq z$. Hence, by Rearrangement and AM-GM we obtain:

$$a^3 b + b^3 c + c^3 a + 3abc = a^2 \cdot ab + b^2 \cdot bc + c^2 \cdot ca + 3abc$$
$$\leq x^2 \cdot xy + y^2 \cdot xz + z^2 \cdot yz + 3xyz = y(x^3 + z^3 + xz(y+3))$$
$$= y((1-y)^3 - 3xz(1-y) + xz(y+3)) = y((1-y)^3 + 4yxz)$$
$$\leq y((1-y)^3 + y(1-y)^2) = y(1-y)^2 = 4y\left(\frac{1-y}{2}\right)^2$$
$$\leq 4\left(\frac{y + \frac{1-y}{2} + \frac{1-y}{2}}{3}\right)^3 = \frac{4}{27}$$

Completing the proof.

**Problem 1.2.49.** For all $x,y,z>0$, prove that

$$\sum_{cyc}\frac{4x+y+z}{x+4y+4z}\geq 2$$

**Solution.** Due to the symmetry, we may assume that $x\geq y\geq z$. Notice that

$$\frac{1}{x+4y+4z}\geq\frac{1}{y+4z+4x}\geq\frac{1}{z+4x+4y}.$$

By Rearrangement inequality, we have

$$\sum_{cyc}\frac{x}{x+4y+4z}\geq\sum_{cyc}\frac{y}{x+4y+4z}\text{ and }\sum_{cyc}\frac{x}{x+4y+4z}\geq\sum_{cyc}\frac{z}{x+4y+4z}.$$

Now, we consider that

$$\begin{aligned}\sum_{cyc}\frac{4x+y+z}{x+4y+4z}-2&=\sum_{cyc}\left(\frac{4x+y+z}{x+4y+4z}-\frac{2}{3}\right)\\&=\frac{5}{3}\sum_{cyc}\frac{2x-y-z}{x+4y+4z}\\&=\frac{5}{3}\left(\sum_{cyc}\frac{x-y}{x+4y+4z}+\sum_{cyc}\frac{x-z}{x+4y+4z}\right)\\&\geq 0.\end{aligned}$$

**Problem 1.2.50** (Balkan MO 2010). Let $a,b$ and $c$ be positive real numbers. Prove that

$$\frac{a^2b(b-c)}{a+b}+\frac{b^2c(c-a)}{b+c}+\frac{c^2a(a-b)}{c+a}\geq 0.$$

**Solution.** Substitution $x=\frac{1}{a},y=\frac{1}{b},z=\frac{1}{c}$, in the equation this inequality become

$$\sum\frac{x-z}{y+z}\geq 0\iff\sum x\left(\frac{1}{y+z}\right)\geq\sum z\left(\frac{1}{y+z}\right)$$

which is true by Rearrangement Inequality.

**Problem 1.2.51** (MMOSL, Aritra12). Prove that the given expression is true where $abc=1$ , $a,b,c>0$,

$$\sum_{cyc}\frac{ac(a+c^5)+1}{b^{12}+a(a^{13}c+1)}\leq a^6+b^6+c^6$$

**Solution.** We know that

$$a^{13}+b^{13}\geq a^{12}b+ab^{12}$$

therefore

$$\frac{ab}{a^{13}+b^{13}+ab}\leq\frac{ab}{a^{12}b+ab^{12}+ab}=\frac{1}{a^{11}+b^{11}+1}$$

Now note that $a^{11}+b^{11}\geq(a+b)(ab)^5$ so its true that ,

$$\frac{1}{a^{11}+b^{11}+1}\leq\frac{1}{(a+b)(ab)^5+1}$$

Also $\frac{1}{(a+b)(ab)^5+1}$ can be written as $abc=1$

$$=\frac{1}{(a+b)(ab)^5+(abc)^5}\implies\frac{1}{(ab)^5\left(a+b+c^5\right)}\implies\frac{(abc)^5}{(ab)^5\left(a+b+c^5\right)}$$

Now $\frac{1}{(ab)^5\left(a+b+c^5\right)}$ is simply $\frac{c^5}{a+b+c^5}$ So we can say that,

$$\frac{abc}{a^{13}+b^{13}+ab}\leq\frac{c^6}{a+b+c^5}$$

$$\frac{a+b+c^5}{a^{13}+b^{13}+ab}\leq c^6$$

so it results that,

$$\frac{a+b+c^5}{a^{13}+b^{13}+ab}+\frac{b+c+a^5}{b^{13}+c^{13}+bc}+\frac{c+a+b^5}{c^{13}+a^{13}+ca}\leq c^6+b^6+a^6=a^6+b^6+c^6$$

$$\sum_{\text{cyc}}\frac{a+b+c^5}{a^{13}+b^{13}+ab}\leq a^6+b^6+c^6$$

Multiplying the first term of LHS with $c$

$$\frac{ac+bc+c^6}{a^{13}c+b^{13}c+abc}\implies\frac{ac+bc+c^6}{a^{13}c+b^{13}c+1}$$

on multiplying with $a$ again

$$\frac{a^2c+abc+ac^6}{a^{14}c+ab^{13}c+a}\implies\frac{a^2c+1+ac^6}{a^{14}c+b^{12}+a}\implies\frac{ac(a+c^5)+1}{b^{12}+a(a^{13}c+1)}\leq c^6$$

similarly the next two terms will be less than $a^6$ and $b^6$ so its true that ,

$$\sum_{cyc}\frac{ac(a+c^5)+1}{b^{12}+a(a^{13}c+1)}\leq a^6+b^6+c^6$$

### 1.2.8 Chebyshev's Inequality

**Theorem 1.2.8** (Chebyshev's Inequality) — If $a_1 \geq a_2 \geq \ldots \geq a_n$ and $b_1 \geq b_2 \geq \ldots \geq b_n$ then the following inequality holds:

$$n\left(\sum_{i=1}^{n} a_i b_i\right) \geq \left(\sum_{i=1}^{n} a_i\right)\left(\sum_{i=1}^{n} b_i\right)$$

On the other hand, if $a_1 \geq a_2 \geq \ldots \geq a_n$ and $b_n \geq b_{n-1} \geq \ldots \geq b_1$ then:

$$n\left(\sum_{i=1}^{n} a_i b_i\right) \leq \left(\sum_{i=1}^{n} a_i\right)\left(\sum_{i=1}^{n} b_i\right)$$

Equality holds if and only if $a_1 = a_1 = \ldots = a_n$ or $b_1 = b_2 = \ldots = b_n$

**Remark.** Chebyshev's Inequality is a consequence of the the Rearrangement inequality. We invite the courageous reader to prove Chebyshev's Inequality.

**Problem 1.2.52.** Prove that if $a,b,c > 0$ with $a+b+c = 1$, then

$$\frac{a^3}{b^2+c^2} + \frac{b^3}{c^2+a^2} + \frac{c^3}{a^2+b^2} \geq \frac{1}{2}$$

**Solution.** WLOG let $a \geq b \geq c$ as the inequality is symmetric. Now we have following ordered sequences arranged having the same order:

$$a \geq b \geq c \text{ and } \frac{a^2}{b^2+c^2} \geq \frac{b^2}{c^2+a^2} \geq \frac{c^2}{a^2+b^2}.$$

Thus by chebyshev inequality, we have,

$$\frac{a^3}{b^2+c^2} + \frac{b^3}{c^2+a^2} + \frac{c^3}{a^2+b^2} \geq \left(\frac{a+b+c}{3}\right) \cdot \left(\frac{a^2}{b^2+c^2} + \frac{b^2}{c^2+a^2} + \frac{c^2}{a^2+b^2}\right).$$

Now, by Nesbitt inequality, we have,

$$\left(\frac{a^2}{b^2+c^2} + \frac{b^2}{c^2+a^2} + \frac{c^2}{a^2+b^2}\right) \geq \frac{3}{2}.$$

So,

$$\left(\frac{a+b+c}{3}\right) \cdot \left(\frac{a^2}{b^2+c^2} + \frac{b^2}{c^2+a^2} + \frac{c^2}{a^2+b^2}\right) \geq \frac{1}{3} \cdot \frac{3}{2} = \frac{1}{2}$$

which is the desired result.

**Problem 1.2.53** (Titu Andreescu). Let $x_1, x_2, \dots, x_n$ be positive real numbers. Prove that

$$n\left(\frac{1}{x_1}+\frac{1}{x_2}+\dots+\frac{1}{x_n}\right) \geq \left(\frac{1}{x_1+1}+\frac{1}{x_2+1}+\dots+\frac{1}{x_n+1}\right)\left(n+\frac{1}{x_1}+\frac{1}{x_2}+\dots+\frac{1}{x_n}\right)$$

**Solution.** Assume WLOG that $x_1 \geq x_2 \geq \dots \geq x_n$. Then, we have

$$\frac{1}{x_i} \leq \frac{1}{x_{i+1}} \implies \frac{1}{x_i+1} \leq \frac{1}{x_{i+1}+1}, \frac{x_{i+1}}{x_i} \leq \frac{x_{i+1}+1}{x_i+1}$$

for any $i \in \{2,3,4,\dots,n-1\}$. According to Chebyshev's Inequality, we have

$$\left(\frac{1}{x_1+1}+\frac{1}{x_2+1}+\dots+\frac{1}{x_n+1}\right)\left(\frac{x_1+1}{x_1}+\frac{x_2+1}{x_2}+\dots+\frac{x_n+1}{x_n}\right) \leq n\left(\frac{1}{x_1}+\frac{1}{x_2}+\dots+\frac{1}{x_n}\right)$$

as desired. Equality holds if and only if $x_1 = x_2 = \dots = x_n$.

**Problem 1.2.54** (French JBMO TST 2019). Let $a, b, c$ be positive real numbers such that $a+b+c=1$. Prove:

$$\frac{5+2b+c^2}{1+a}+\frac{5+2c+a^2}{1+b}+\frac{5+2a+b^2}{1+c} \geq 13.$$

**Solution.**

$$\frac{5+2b+c^2}{1+a}+\frac{5+2c+a^2}{1+b}+\frac{5+2a+b^2}{1+c} = 5\sum_{cyc}\frac{1}{1+a}+2\sum_{cyc}\frac{b}{1+a}+\sum_{cyc}\frac{c^2}{1+a}$$

Now by cauchy's extended inequality and the fact that a+b+c=1

$$\sum_{cyc}\frac{1}{1+a} \geq \frac{9}{4}$$

$$\sum_{cyc}\frac{c^2}{1+a} \geq \frac{1}{1+a}$$

$$\sum_{cyc}\frac{b}{1+a} = \sum_{cyc}\frac{b^2}{b+ab} \geq \frac{1}{1+\sum_{cyc} ab}$$

Also by chebyshev's inequality ,

$$\sum_{cyc} ab \leq \frac{(a+b+c)(b+c+a)}{3}$$

From the above three inequalities, we get the desired inequality.

**Problem 1.2.55** (INMO 2020 P4). Let $n \geqslant 2$ be an integer and let $1 < a_1 \leq a_2 \leq \cdots \leq a_n$ be $n$ real numbers such that $a_1 + a_2 + \cdots + a_n = 2n$. Prove that

$$a_1a_2 \ldots a_{n-1} + a_1a_2 \ldots a_{n-2} + \cdots + a_1a_2 + a_1 + 2 \leqslant a_1a_2 \ldots a_n.$$

**Solution.** For convenience, let $P_i = a_1a_2 \ldots a_i$. Apply Chevbyshev's inequality on $1 < P_1 < P_2 < \ldots < P_{n-1}$ and $a_1 \leq a_2 \leq \ldots \leq a_n$ gives

$$\begin{aligned}(P_1 + P_2 + \ldots + P_n) &\geq \frac{1}{n}(a_1 + a_2 + \ldots + a_n)(1 + P_1 + P_2 + \ldots + P_{n-1}) \\ &= 2(1 + P_1 + P_2 + \ldots + P_{n-1})\end{aligned}$$

which a simple rearranging makes this equivalent to the conclusion. Equality holds if and only if $a_1 = a_2 = \cdots = a_n = 2$.

**Problem 1.2.56** (Vasile Cirtoaje). Let $a_1, a_2, ..., a_n$ be nonnegative reals such that $a_1 + a_2 + ... + a_n = n$. Prove that

$$(n+1)\left(a_1^2 + a_2^2 + ... + a_n^2\right) \geq n^2 + a_1^3 + a_2^3 + ... + a_n^3$$

**Solution.** Assume WLOG that $a_1 \geq a_2 \geq ... \geq a_n$. Replacing $n^2$ with

$$n\left(a_1 + a_2 + ... + a_n,\right)$$

the desired inequality rewrites as

$$\sum_{i=1}^{n}\left[(n+1)a_i^2 - na_i - a_i^3\right] \geq 0$$

or

$$\sum_{i=1}^{n}(a_i - 1)\left(na_i - a_i^2\right) \geq 0.$$

Since

$$a_1 - 1 \geq a_2 - 1 ... \geq a_n - 1,$$

and since $\left(na_i - a_i^2\right) - \left(na_j - a_j^2\right) \geq 0$, we have

$$na_1 - a_1^2 \geq na_2 - a_2^2 ... \geq na_n - a_n^2.$$

We apply Chebyshev's Inequality to get

$$n\sum_{i=1}^{n}(a_i - 1)\left(na_i - a_i^2\right) \geq \left[\sum_{i=1}^{n}(a_i - 1)\right]\left[\sum_{i=1}^{n}(na_i - a_i^2)\right]$$

as desired. Equality holds if and only if $a_1 = a_2 = ... = a_n = 1$.

**Problem 1.2.57** (Romania TST 2006). Let $a, b, c$ be positive real numbers such that $a+b+c=3$. Prove that:

$$\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2} \geq a^2+b^2+c^2.$$

**Solution.** Rewrite the inequality in the form

$$\sum_{cyc} a^2b^2 \geq a^2b^2c^2 \sum_{cyc} a^2 \iff \sum_{cyc} a^2b^2\left(1+c+c^2+c^3\right)(1-c) \geq 0.$$

Notice that if $ab \leq 2$ and $a \geq b$ then

$$a^2(1+b+b^2+b^3) \geq b^2(1+a+a^2+a^3).$$

This one is equivalent to

$$(a-b)(a+b+ab-a^2b^2) \geq 0$$

which is obviously true because $ab \leq 2$. From this property, we conclude that if all $ab, bc, ca$ are smaller than 2 then Chebyshev inequality yields

$$\sum_{cyc} a^2b^2\left(1+c+c^2+c^3\right)(1-c) \geq \left(\sum_{sym} a^2b^2\left(1+c+c^2+c^3\right)\right)\left(\sum_{sym}(1-c)\right).$$

Otherwise $ab \geq 2$. Clearly, $a+b \geq 2\sqrt{2}$, so $c \leq 3-2\sqrt{2}$ and $c^2 \leq \frac{1}{9}$. That means

$$\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2} \geq 9 \geq a^2+b^2+c^2.$$

The proof is finished. Equality holds for $a=b=c=1$.

## 1.2.9 Minkowski's Inequality

**Theorem 1.2.9** (Minkowski's Inequality) — Let $r > s$ be nonzero real numbers, then for any sequence of positive reals $\{a_{ij}\}^n_{1\leq i<j\leq n}$, the following inequality holds:

$$\left(\sum_{j=1}^{m}\left(\sum_{i=1}^{n} a_{ij}^r\right)^{s/r}\right)^{1/s} \geq \left(\sum_{i=1}^{n}\left(\sum_{j=1}^{m} a_{ij}^s\right)^{r/s}\right)^{1/r}$$

Note that if either $r$ or $s$ is zero, then Minkowski's Inequality becomes Holder's Inequality.

**Remark.** The Holder's Inequality is a consequence of Minkowski's Inequality. The proofs are similar and different in different aspects. We invite the courageous reader to prove both Holder's and Minkowski's and find the similarity and differences.

**Problem 1.2.58.** Given $a,b,c>0$, find the minimum value of

$$\sqrt{(a-x)^2+y^2+z^2}+\sqrt{x^2+(b-y)^2+z^2}+\sqrt{x^2+y^2+(c-z)^2}+\sqrt{(a-x)^2+(b-y)^2+(c-z)^2}$$

Where $x,y,z$ are real numbers.

> **Solution.** By Minkowski's Inequality, the answer turns out to be: $2\sqrt{a^2+b^2+c^2}$

**Problem 1.2.59** (AIME 1991). For positive integer $n$, define $S_n$ to be the minimum value of the sum

$$\sum_{k=1}^{n}\sqrt{(2k-1)^2+a_k^2},$$

where $a_1,a_2,\ldots,a_n$ are positive real numbers whose sum is 17. There is a unique positive integer $n$ for which $S_n$ is also an integer. Find $n$.

> **Solution.** By Minkowski's Inequality, we have
>
> $$S_n \geq \sqrt{\left(\sum_{k=1}^{n}(2k-1)\right)^2+\left(\sum_{k=1}^{n}a_k\right)^2}$$
>
> or
>
> $$S_n \geq \sqrt{n^4+289}\in\mathbb{Z}$$
>
> So, there exists an integer $m$ for which $n^4+289=m^2$, or
>
> $$(n^2+m)(n^2-m)=289$$
>
> we find that $n=12$ and $m=145$, revealing the value of $n$ as 12

**Problem 1.2.60** (OMO Fall 2013). Let $ABCD$ be a quadrilateral with $AD=20$ and $BC=13$. The area of $\triangle ABC$ is 338 and the area of $\triangle DBC$ is 212. Compute the smallest possible perimeter of $ABCD$.

**Solution.** Let $B=(0,0)$ and $C=(13,0)$. Then $A=(a,52)$ and $D=\left(b,\frac{424}{13}\right)$ for some $a$ and $b$. Now it remains to minimize the function

$$f(a,b)=\sqrt{a^2+52^2}+\sqrt{(b-13)^2+\left(\frac{424}{13}\right)^2}$$

subject to $(b-a)^2+\left(\frac{252}{13}\right)^2=20^2$, or $b-a=\frac{64}{13}$. Substituting $b=\frac{64}{13}+a$ shows that we have to minimize

$$f(a)=\sqrt{a^2+52^2}+\sqrt{\left(a-\frac{105}{13}\right)^2+\left(\frac{424}{13}\right)^2}.$$

Using Minkowski's we get

$$f(a)=\sqrt{a^2+52^2}+\sqrt{\left(\frac{105}{13}-a\right)^2+\left(\frac{424}{13}\right)^2}\geq\sqrt{\left(\frac{105}{13}\right)^2+\left(52+\frac{424}{13}\right)^2}=85.$$

Thus the minimum perimeter is $85+20+13=\boxed{118}$.

**Problem 1.2.61** (Moldova TST 2014). Let $a,b\in\mathbb{R}_+$ such that $a+b=1$. Find the minimum value of the following expression:

$$E(a,b)=3\sqrt{1+2a^2}+2\sqrt{40+9b^2}.$$

**Solution.** Write inequality as:

$$\sqrt{1+a^2+a^2}+\sqrt{1+a^2+a^2}+\sqrt{1+a^2+a^2}+\sqrt{6^2+2^2+9b^2}+\sqrt{6^2+2^2+9b^2}$$

And apply Minkowski Inequality:

$$E(a,b)\geq\sqrt{(1+1+1+6+6)^2+(a+a+a+2+2)^2+(a+a+a+3b+3b)^2}$$
$$=\sqrt{225+(3a+4)^2+(3a-6)^2}=\sqrt{277+6(3a^2-2a)}$$

Define function $f:(0,1]\to R,\ f(x)=3x^2-2x$. Then we get $f'(x)=6x-2$. For $x\in\left(0;\frac{1}{3}\right]$ the funtion is decreasing, and for $x\in\left[\frac{1}{3},1\right]$ the function is increasing. So minimum value of the function is $f\left(\frac{1}{3}\right)=-\frac{1}{3}$. So we get:

$$E(a,b)\geq\sqrt{277+6(3a^2-2a)}\geq\sqrt{277-2}=5\sqrt{11}.$$

Equality occurs iff $a=\frac{1}{3},b=\frac{2}{3}$.

**Problem 1.2.62** (Baltic Way 2000). Prove that for all positive real numbers $a,b,c$ we have

$$\sqrt{a^2-ab+b^2}+\sqrt{b^2-bc+c^2}\geq\sqrt{a^2+ac+c^2}$$

**Solution.** Using Minkowski's inequality:

$$\sqrt{\left(\frac{a\sqrt{3}}{2}\right)^2+\left(\frac{a}{2}-b\right)^2}+\sqrt{\left(\frac{c\sqrt{3}}{2}\right)^2+\left(b-\frac{c}{2}\right)^2}\geq\sqrt{\left(\frac{a}{2}-\frac{c}{2}\right)^2+\frac{3}{4}(a+c)^2}=\sqrt{a^2+ca+c^2}.$$

**Problem 1.2.63.** Let $a,b,c,d$ be positive real numbers then prove that

$$\sum_{cyc}\sqrt{a^2+b^2-ab\sqrt{3}}\geq a\sqrt{3}.$$

Equality holds if and only if

$$a=b\sqrt{3}=2c=d\sqrt{3}.$$

**Solution.** Using Minkowski's inequality:

$$\sqrt{a^2-ab\sqrt{3}+b^2}+\sqrt{b^2-bc\sqrt{3}+c^2}=\sqrt{\frac{a^2}{4}+\left(\frac{a\sqrt{3}}{2}-b\right)^2}+\sqrt{\frac{c^2}{4}+\left(b-\frac{c\sqrt{3}}{2}\right)^2}$$

$$\sqrt{a^2-ab\sqrt{3}+b^2}+\sqrt{b^2-bc\sqrt{3}+c^2}\geq\sqrt{\left(\frac{a+c}{2}\right)^2+\left(\frac{a\sqrt{3}-c\sqrt{3}}{2}\right)^2}$$

$$\sqrt{a^2-ab\sqrt{3}+b^2}+\sqrt{b^2-bc\sqrt{3}+c^2}\geq\sqrt{\frac{3a^2}{4}+\left(\frac{a}{2}-c\right)^2}\geq\frac{a\sqrt{3}}{2}\quad(1)$$

$$\text{Equality when and only if:}\quad \frac{a}{2}-c=0,\quad \frac{a}{c}=\frac{a\sqrt{3}-2b}{2b-c\sqrt{3}}\iff a=2c,\quad a=b\sqrt{3}$$

$$\sqrt{c^2-cd\sqrt{3}+d^2}+\sqrt{d^2-da\sqrt{3}+a^2}=\sqrt{\frac{c^2}{4}+\left(\frac{c\sqrt{3}}{2}-d\right)^2}+\sqrt{\frac{a^2}{4}+\left(d-\frac{a\sqrt{3}}{2}\right)^2}$$

$$\sqrt{c^2-cd\sqrt{3}+d^2}+\sqrt{d^2-da\sqrt{3}+a^2}\geq\sqrt{\left(\frac{c+a}{2}\right)^2+\left(\frac{c\sqrt{3}-a\sqrt{3}}{2}\right)^2}$$

$$\sqrt{c^2-cd\sqrt{3}+d^2}+\sqrt{d^2-da\sqrt{3}+a^2}\geq\sqrt{\frac{3a^2}{4}+\left(\frac{a}{2}-c\right)^2}\geq\frac{a\sqrt{3}}{2}\quad(2)$$

$$\text{Equality when and only if:}\quad a=2c,\quad a=d\sqrt{3}$$

Thus $(1)+(2)$ gives

$$\sum\sqrt{a^2-ab\sqrt{3}+b^2}\geq a\sqrt{3},$$

and equality holds if and only if $a=b\sqrt{3}=2c=d\sqrt{3}$.

## 1.2.10 Practice Problems

We have now come to the end of the handout, and we therefore conclude with a set of practice problems.

**Practice Problems.**

1. Let $0\leq a\leq b\leq c\leq d$ such that $ab+bc+cd+da+ac+ad=6$.

(i) Find the maximum of $abcd(a+c)$

(ii) Find the maximum of $abcd(a+c)^8$

2. Let $a\geq b\geq c>0$ and $abc=1$. Prove or disprove that

$$a+c^2\geq\frac{5\sqrt[5]{4}}{4}b$$

3. Let $a,b,c\geq 0$ and $ab+bc+ca+abc=1$. Prove that

$$\frac{a}{a+bc}+\frac{b}{b+ca}+\frac{c}{c+ab}\leq 2$$

4. Let $a,b,c\geq 0$. Prove that

$$\frac{a^2}{2a^2+bc}+\frac{b^2}{2b^2+ca}+\frac{c^2}{2c^2+ab}\leq 1$$

5. Let $a\geq 2,b\geq 2$. Prove that

$$\frac{a}{a+b^2+1}+\frac{b}{b+a^2+1}\leq\frac{4}{7}.$$

6. Let $a,b$ be nonnegative reals such that $a+b\leq 2$.Prove that

$$(1+a^2)(1+b^2)\geq\left[1+\left(\frac{a+b}{2}\right)^2\right]^2$$

7.(Mathlinks) Let $a, b, c$ be nonnegative reals such that $a+b+c=2$. Prove that

$$(a^3+b^3)(b^3+c^3)(c^3+a^3) \leq 2$$

8.(Mathlinks) Let $a_1, a_2, ..., a_n$ be positive real numbers. Prove that

$$\sum_{i=1}^{n} \frac{a_{i+1}}{a_i} \geq \sum_{i=1}^{n} \sqrt{\frac{a_{i+1}^2+1}{a_i^2+1}}$$

where $a_{n+1} = a_1$

9.(Mathlinks) Let $x, y, z$ be nonnegative real numbers. Prove that

$$(x+2y+3z)(x^2+y^2+z^2) \geq \frac{20-2\sqrt{2}}{27}(x+y+z)^3$$

When does equality hold?

10. Prove that for all reals $a, b, c, d, e$,

$$2a^2+b^2+3c^2+d^2+2e^2 \geq 2(ab-bc-cd-de+ea)$$

When does equality hold?

11. Let $a, b, c$ be nonnegative real numbers such that $a+b+c=1$. Prove that

$$4 \geq (3a^2+1)(3b^2+1)(3c^2+1) \geq \frac{64}{27}$$

12.(Rama1728) Let $a, b, c$ be reals such that $abc=1$. Prove that

$$\sum_{cyc} \frac{a^3}{bc+ab} \geq \frac{a+b+c}{2}$$

13. Let $a, b, c \geq -1$ and $a^3+b^3+c^3=1$ . Prove that

$$a+b+c+a^2+b^2+c^2 \leq 4.$$

When does the equality hold ?

13.(USA TST 2010) Let $a, b, c$ be positive real numbers such that $abc=1$. Show that

$$\frac{1}{a^5(b+2c)^2}+\frac{1}{b^5(c+2a)^2}+\frac{1}{c^5(a+2b)^2} \geq \frac{1}{3}$$

14.(Mathematics and Youth Magazine) Given positive real numbers $x, y, z$ satisfying

$$x^2 + y^2 - 2z^2 + 2xy + yz + zx \leq 0,$$

find the minimum of

$$P = \frac{x^4 + y^4}{z^4} + \frac{y^4 + z^4}{x^4} + \frac{z^4 + x^4}{y^4}$$

15. Let $a, b, c$ be the side-lengths of a triangle. If $0 < r \leq 1$, then

$$a^2 b\left(a^r - b^r\right) + b^2 c\left(b^r - c^r\right) + c^2 a\left(c^r - a^r\right) \geq 0$$

16. If $a_i > 0 (i = 1, 2, \ldots, n)$ and

$$L(k) = \left(\sum_{cyclic} \frac{a_1^k}{a_1^{k-1} + a_2^{k-1}}\right)\left(\sum_{cyclic} \frac{a_2^{k-1}}{a_1\left(a_1^{k-1} + a_2^{k-1}\right)}\right), \text{ then}$$

prove that

$$L(k) \geq L(k-1) \geq .. \geq L(0)$$

17.(Aritra12) Prove that the given inequality is true where $abc = 1$ , $a, b, c > 0$,

$$\sum_{cyc} \frac{ac(a + c^5) + 1}{b^{12} + a(a^{13}c + 1)} \leq a^6 + b^6 + c^6$$

18. Let $a, b, c, d, e, f$ be positive real numbers. Prove the inequality

$$\frac{a}{b+c} + \frac{b}{c+d} + \frac{c}{d+e} + \frac{d}{e+f} + \frac{e}{f+a} + \frac{f}{a+b} \geq 3$$

19. Prove that for real numbers $x_1, x_2, x_3, y_1, y_2, y_3$ such that

$$x_1x_2 + x_2x_3 + x_3x_1 \geq 0$$

and

$$y_1y_2 + y_2y_3 + y_3y_1 \geq 0$$

the following inequality holds.

$$(y_2 + y_3)x_1 + (y_3 + y_1)x_2 + (y_1 + y_2)x_3 \geq 2\sqrt{(x_1x_2 + x_2x_3 + x_3x_1)(y_1y_2 + y_2y_3 + y_3y_1)}$$

Also find if equality holds ?

20.(Involving some FE) Find all functions $f:\mathbb{N}\to\mathbb{N}$ with $f(1)=1$ and

$$f(x+5)\geq f(x)+5 \text{ and } f(x+1)\leq f(x)+1$$

holds for all $x\in\mathbb{N}$

21. Let $a,b,c$ be positive real numbers such that $a+b+c=3$. Prove that:

$$\sqrt{3a+\frac{1}{b}}+\sqrt{3b+\frac{1}{c}}+\sqrt{3c+\frac{1}{a}}\geq 6$$

22. Let $a,b,c$ be positive real numbers. Prove that

$$\sum_{cyc}\sqrt{(a+b-1)^2+2c^2}\geq\sqrt{3}$$

23.(Grotex) Let $a,b,c$ be non-negative real numbers. Prove that

$$\sqrt{a^3+2}+\sqrt{b^3+2}+\sqrt{c^3+2}\geq\sqrt{\frac{9+3\sqrt{3}}{2}(a^2+b^2+c^2)}.$$

24.(Kunihiko Chikaya) Let $a,\ b,\ c$ be positive real numbers. Prove that

$$\frac{\sqrt{a+b+c}}{\sqrt{ab}+\sqrt{bc}+\sqrt{ca}}\left(\sqrt{\frac{b^2}{a}+\frac{c^2}{b}}+\sqrt{\frac{c^2}{b}+\frac{a^2}{c}}+\sqrt{\frac{a^2}{c}+\frac{b^2}{a}}\right)\geq\sqrt{6}.$$

25.(sqing) Let $a,b$ be positive real numbers such that $a+b=1$. Prove that

$$a\sqrt{b^2+1}+b\sqrt{a^2+1}\geq\sqrt{20a^2b^2+(a-b)^2}.$$

26.(sqing) Let $a,b,c,d$ be positive real numbers . Prove that

$$\frac{ab}{c+d}+\frac{bc}{d+a}+\frac{cd}{a+b}+\frac{da}{b+c}+\frac{ac}{b+d}+\frac{bd}{c+a}\geq\frac{3}{4}(a+b+c+d).$$

27.(sqing) Let $a,b$ be positive numbers such that $ab\geq 1$. Prove that

$$a\sqrt{b+1}+b\sqrt{a+1}+\sqrt{a+b}\geq 3\sqrt{2}.$$

28.(sqing) Let $a,b$ be positive real numbers such that $a+b=1$. Prove that

$$\text{(a)}\sqrt{20a^2b^2+(a-b)^2}\leq a\sqrt{1+b^2}+b\sqrt{1+a^2}\leq 4\sqrt{5}ab+(a-b)^2$$

$$\text{(b)}\sqrt{12a^2b^2+\frac{1}{4}(a-b)^2}\leq a\sqrt{1-b^2}+b\sqrt{1-a^2}\leq 2\sqrt{3}ab+\frac{1}{4}(a-b)^2.$$

30.(S. Lenny) Let $a, b$ be nonnegative real numbers such that $a+b=2$. Prove that

$$4a\sqrt{b+1}+4b\sqrt{a+1}+\sqrt{2}\geq 9\sqrt{ab+1}$$

31.(sqing) Let $a, b$ be positive numbers such that $a+b=2$. Prove that

$$\text{(a) } (a+\sqrt{b^2+1})(b+\sqrt{a^2+1})\geq 3+2\sqrt{2}$$

$$\text{(b) } a(\sqrt{b^2+1}-b)+b(\sqrt{a^2+1}-a)\geq 2(\sqrt{2}-1)$$

32.(man11) If $a, b, c\in\mathbb{R}$ and $a+b+c=2$.Then find the minimum of

$$\sqrt{a^2+7}+\sqrt{b^2+11}+\sqrt{c^2+13}$$

33.(a-simple-guy) Let $a, b, c$ be positive real numbers with sum 3. Prove that:

$$\sqrt{a^2+\sqrt{b^2+c^2}}+\sqrt{b^2+\sqrt{c^2+a^2}}+\sqrt{c^2+\sqrt{a^2+b^2}}\geq 3\sqrt{\sqrt{2}-1}$$

34.(CSS-MU) Let $a, b, c$ be real numbers satisfying $a+b+c=3$. Show that

$$\frac{1}{2a^2+7}+\frac{1}{2b^2+7}+\frac{1}{2c^2+7}\leq\frac{1}{3}$$

35.(danciulian) Let $a\geq c\geq 0$ and $b\geq d\geq 0$. Prove that

$$(a+b+c+d)^2\geq 8(ad+bc).$$

36.(Samin Riasat) Let $a_1, a_2, ..., a_n\in[0,1]$ and $\lambda$ be real number such that $a_1+a_2+...+a_n=n+1-\lambda$. For any permutation $(b_i)_{i=1}^n$ of $(a_i)_{i=1}^n$, prove that

$$a_1b_1+a_2b_2+...+a_nb_n\geq n+1-\lambda^2$$

. 37.(Known Result) Given that $a, b, c$ are positive real numbers, and $abc=1$. Prove that , for any natural number $k\geq 2$

$$\frac{a^k}{a+b}+\frac{b^k}{b+c}+\frac{c^k}{c+a}\geq\frac{3}{2}$$

38.(OIM/IOM) Find all positive real solutions of the system of equations

$$\begin{cases} x_1+x_2+...+x_{1994}=1994 \\ x_1^4+x_2^4+...+x_{1994}^4=x_1^3+x_2^3+....+x_{1994}^3 \end{cases}$$

39.(Kyiv Mathematical festival) Let $a, b, c\geq 0$ and $a+b+c\geq 3$. Prove that

$$a^4+b^3+c^2\geq a^3+b^2+c$$

40.(Smathematician) Let $a,b,c \in \mathbb{R}^+$ and $n \geq 2$ is an integer. Prove:-

$$\frac{a^n}{b+c}+\frac{b^n}{c+a}+\frac{c^n}{a+b} \geq \left(\frac{2}{3}\right)^{(n-2)}\left(\frac{a+b+c}{2}\right)^{(n-1)}$$

41.(lahmacun) Let $a_1, a_2, \ldots, a_n$ be distinct positive integers. Prove that

$$\frac{a_1}{1^2}+\frac{a_2}{2^2}+\cdots+\frac{a_n}{n^2} \geq \frac{1}{1}+\frac{1}{2}+\cdots+\frac{1}{n}$$

42.(Outwitter) For $a,b,c \in \mathbb{R}^+$ and $abc=1$. Prove that

$$\frac{c}{b}+\frac{b}{a}+\frac{a}{c} \leq a^3b+b^3c+c^3a$$

43.(Didier) Let $a,b,c$ be the sides of an acute triangle. Prove that

$$\sqrt{a^2+b^2-c^2}+\sqrt{a^2-b^2+c^2}+\sqrt{b^2+c^2-a^2} \leq \sqrt{3(ab+bc+ca)}$$

44.(2013 China TST) Let $k \geq 2$ be an integer and let $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n$ be nonnegative real numbers. Prove that

$$\left(\frac{n}{n-1}\right)^{n-1}\left(\frac{1}{n}\sum_{i=1}^{n} a_i^2\right)+\left(\frac{1}{n}\sum_{i=1}^{n} b_i\right)^2 \geq \prod_{i=1}^{n}(a_i^2+b_i^2)^{\frac{1}{n}}$$

45.(Lonesan) For any triangle $ABC$, prove that:

$$s \geq 3\sqrt{3}r+\frac{1}{2}\sqrt{\frac{(s^2-12Rr-3r^2)(R-2r)}{R}}$$

46.(USAMO 2000) Let $a_1, b_1, a_2, b_2, \ldots, a_n, b_n$ be nonnegative real numbers. Prove that

$$\sum_{i,j=1}^{n} \min\{a_ia_j, b_ib_j\} \leq \sum_{i,j=1}^{n} \min\{a_ib_j, a_jb_i\}.$$

47.(CCNHSMO 2012) Let $a$, $b$, $c$ be positive real numbers . Prove that

$$\left(a^3+\frac{1}{b^3}-1\right)\left(b^3+\frac{1}{c^3}-1\right)\left(c^3+\frac{1}{a^3}-1\right) \leq \left(abc+\frac{1}{abc}-1\right)^3$$

48.(sqing) Let $a \geq 4, b, c \geq 0, a+b \leq 2c, x, y, z \in R$. Prove that

$$(a-3)(b-x^2-y^2-z^2) \leq (c-x-y-z)^2.$$

49.(CSEMO 2020) Let $0 \le a_1 \le a_2 \le \cdots \le a_{n-1} \le a_n$ and $a_1 + a_2 + \cdots + a_n = 1$. Prove that: For any non-negative numbers $x_1, x_2, \cdots, x_n; y_1, y_2, \cdots, y_n$ , have

$$\left(\sum_{i=1}^{n} a_i x_i - \prod_{i=1}^{n} x_i^{a_i}\right)\left(\sum_{i=1}^{n} a_i y_i - \prod_{i=1}^{n} y_i^{a_i}\right) \le a_n^2 \left(n\sqrt{\sum_{i=1}^{n} x_i \sum_{i=1}^{n} y_i} - \sum_{i=1}^{n} \sqrt{x_i} \sum_{i=1}^{n} \sqrt{y_i}\right)^2 .$$

50.(Problems from the Book) Let $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n$ be real numbers such that

$$\sum_{1 \le i < j \le n} a_i a_j > 0.$$

Prove the inequality

$$\left(\sum_{1 \le i \ne j \le n} a_i b_j\right)^2 \ge \left(\sum_{1 \le i \ne j \le n} a_i a_j\right)\left(\sum_{1 \le i \ne j \le n} b_i b_j\right)$$

## 1.3 References

[1] 116 Algebraic Inequalities from the AMY

[2] 109 Inequalities from the AMSP

[3] AoPS : artofproblemsolving.com

[4] Inequalities problem by Titu Andresscu

[5] Inequalities all around the world